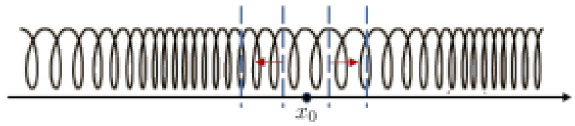
Problem 1.

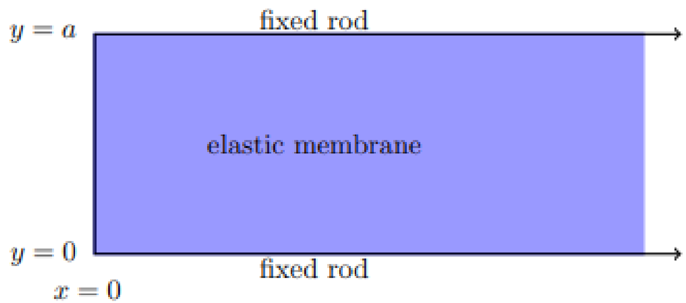


Consider a massive spring which at equilibrium has mass density ρ (per unit length) and Hooke's constant per unit length μ. If we push on one end of the spring, there can be longitudinal vibrations along the spring as shown in the figure. Let's try to establish the equation of motion for the spring. We use ψ to denote longitudinal displacement along the spring, x the equilibrium spatial location, and t the time.

What is the dispersion relation of longitudinal wave in the massive spring?

Problem 2.

Two-dimensional elastic membrane. A uniform semi-infinite membrane is stretched in the $z = 0$ plane, as shown in the figure below.



It is attached to fixed rods along $y = 0$, $z = 0$ and $y = a$, $z = 0$ from $x = 0$ to ∞. $\psi(x, y, t)$ is the z displacement of the point on the membrane with equilibrium position $(x, y, 0)$. For small oscillations, ψ satisfies the two-dimensional wave equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi.$$

a) If this system is extended to an infinite system by continuing it to negative x, show that the following equation gives normal modes of the system:

$$\psi(x, y) = A\sin(nk_0 y)e^{ikx},$$

where n is an integer. That is, show that

$$\psi(x, y, t) = A\sin(nk_0 y)e^{ikx}e^{i\omega t}$$

is a solution to the wave equation and determine the relation between ω and nk_0, k. Find the smallest k_0 that satisfies the boundary condition at $y = 0$ and $y = a$.

b) Back to the semi-infinite membrane. Suppose that the end of the membrane at $x = 0$ is driven as follows:

$$\psi(0, y, t) = [B\sin(3k_0 y) + C\sin(13k_0 y)]\cos(5vk_0 t).$$

The boundary condition at $x = \infty$ is such that there is no wave traveling in the $-x$ direction along the membrane. Find $\psi(x, y, t)$ in steady state.

c) Explain the following statement: For $\omega < 2vk_0$, the system acts like a one-dimensional wave carrier (i.e. the y direction decouples from the x and t directions in the wave equation) with the dispersion relation $\omega^2 = v^2k^2 + \omega_0^2$. What is ω_0?

Problem 3.

Consider two transmission lines with impedance z_1 and z_2. When the two lines are joined together, the boundary condition at their connection point is such that the total voltage and current on the left hand side are equal to the total voltage and current on the right hand side. Suppose that we send a traveling wave from left to right.

a) Write down the general form of the incoming wave, reflected wave and transmitted wave with their amplitude as free parameters to be determined from boundary condition. Specify the frequency and wave number for each of the waves.

b) Use the boundary conditions to find the ratio between the amplitudes of the reflected wave and the incoming wave, and between the transmitted wave and the incoming wave.

Solution 1

For dispersion relations, we essentially need two things,

i) mass

ii) forces acting on that mass

For a small segment of length Δx,

$$m = \rho \Delta x$$

Let $\psi(x)$ be the displacement at point x, then the change in length over Δx is,

$$\Delta L = \psi(x_0 + \Delta x) - \psi(x_0)$$

$$= \left.\frac{\partial \psi}{\partial x}\right|_{x_0} \Delta x \quad \ldots \text{ Ignore higher order.}$$

Similar to Young's modulus,

$$K = \frac{EA}{a} = \frac{\mu}{a} = \frac{\mu}{\Delta x/2}$$

$$\therefore \quad F_L = \text{[illegible]}$$

$$= K\left[\psi(x_0) - \psi(x_0 - \Delta x/2)\right]$$

$$= \frac{\mu}{\Delta x/2}\left[\frac{\partial \psi}{\partial x}\frac{\Delta x}{2}\right] = \mu \left.\frac{\partial \psi}{\partial x}\right|_{x_0 - \Delta x/2}$$

$$\therefore F_R = -\mu \left.\frac{\partial \psi}{\partial x}\right|_{x_0 + \frac{\Delta x}{2}}$$

$\therefore$ Eqns of motion,

$$\rho \Delta x \frac{\partial^2 \psi}{\partial t^2} = \mu \left.\frac{\partial \psi}{\partial x}\right|_{x_0 - \Delta x/2} - \mu \left.\frac{\partial \psi}{\partial x}\right|_{x_0 + \Delta x/2}$$

$$\rho \frac{\partial^2 \varphi}{\partial t^2} = \mu \frac{\partial^2 \varphi}{\partial x^2}$$

$\therefore$ using $\varphi = Ae^{i(kx-\omega t)}$

$$\Rightarrow \rho(-\omega^2) = \mu(-k^2)$$

$$\Rightarrow \omega^2 = \frac{\mu}{\rho} k^2$$

non ω dependent wave speed.
i.e. it is non dispersive.

Solution 2

a) Just plug in to check.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{V^2}\frac{\partial^2 \psi}{\partial t^2}$$

$$\Rightarrow -k^2 \psi + (-n^2 K_0^2)\psi = \frac{1}{V^2}(-\omega^2)$$

$$\Rightarrow \omega^2 = V^2 (k^2 + (nK_0)^2)$$

$A \sin(0) = 0$

$A \sin(n K_0 a) = 0$ mode number

$n K_0 a = m\pi$

$K_0 = \frac{\pi}{a}$

b) We can solve each variable separately

$X(x) = X_0 e^{ikx}$ $\begin{cases} x = 0, \text{ finite value} \\ x = \infty, 0 \end{cases}$ in order to have no wave in the $-x$ direct

$$\therefore X(x) = X_0 e^{-\alpha x}$$

$$Y(y) = D \sin(3K_0 y) + E \sin(13 K_0 y)$$

$$T(t) = \cos(vK_0 t)$$

$$\therefore \psi(x, y, t) = \psi_0 \left[e^{-\alpha x}\right]\left[D\sin(3K_0 y) + E\sin(13K_0 y)\right]\cos(vK_0 t)$$

$v^2(k^2 + (nk_0)^2) < 4v^2 k_0^2$

$\hookrightarrow$ smallest $n = 3$

$\Rightarrow k^2 < -5k_0^2$

$\therefore$ k is imaginary or in other words we have a decaying soln in the x direction.

$\omega_0^2 = 9v^2 k_0^2$

$\Rightarrow \omega_0 = 3vk_0$

Solution 3

a) $V(x,t) = V_+ e^{j(\omega t - kx)} + V_- e^{j(\omega t + kx)}$

$I(x,t) = \frac{V_+}{Z_1} e^{j(\omega t - kx)} - \frac{V_-}{Z_1} e^{j(\omega t + kx)}$ $\Big\}$ incident & reflected

$V'(x,t) = V_t e^{j(\omega t - k'x)}$

$I'(x,t) = \frac{V_t}{Z_2} e^{j(\omega t - k'x)}$

b) At $x = 0$,

$$V_+ + V_- = V_t$$

$$\frac{V_+}{Z_1} - \frac{V_-}{Z_1} = \frac{V_t}{Z_2}$$

$$V_t = \frac{2Z_2}{Z_1 + Z_2} V_+$$

$$V_- = \frac{Z_2 - Z_1}{Z_1 + Z_2} V_+$$