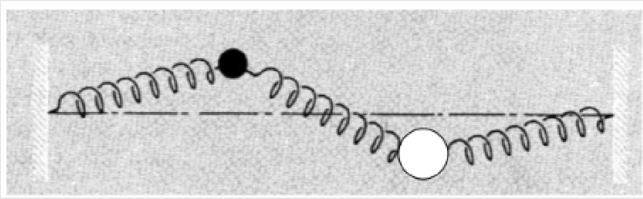
Problem 1
Solve the eigenstates and eigenvalues for the transversal modes for two couple oscillators of mass m1 and m2 in the approximation of small angles. k_{12} is the spring constant between the two masses. The other springs have spring constant k. The masses are m_1 and m_2. When calculating eigenvalues, feel free to consider all k's and m's as equal.



Solution:

$$\begin{pmatrix} m_1\ddot{x_1} \\ m_2\ddot{x_2} \end{pmatrix} = \begin{pmatrix} -kx_1 + k_{12}(x_2 - x_1) \\ -kx_2 - k_{12}(x_2 - x_1) \end{pmatrix}$$

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x_1} \\ \ddot{x_2} \end{pmatrix} = - \begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

You can solve it the usual way but you also learned the following in the HW:

$$(M^{-1/2}KM^{-1/2} - \omega^2 I)y = 0, \text{ with } y = M^{1/2}x$$

If you take $k_{12} = k$ and $m_1 = m_2 = m$,

$$\omega = \sqrt{\frac{3k}{m}}, \ \sqrt{\frac{k}{m}}$$

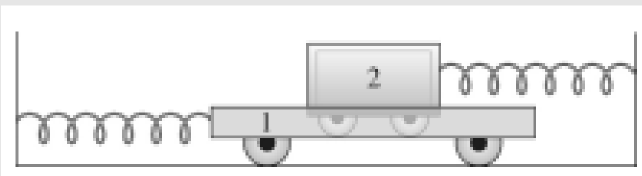
and the regular symmetric and antisymmetric eigenvectors.

The general solution is,

$$\omega = \sqrt{\frac{k + k_{12}}{2m_1m_2} \mp \frac{1}{2}\sqrt{k^2(\frac{1}{m_2} - \frac{1}{m_1})^2 + 2kk_{12}(\frac{1}{m_2} - \frac{1}{m_1})^2 + k_{12}^2(\frac{1}{m_2} + \frac{1}{m_1})^2}}$$

Left out the eigenvectors as they look huge.

Problem 2
In this problem, we tackle a coupled oscillator (two degrees of freedom) with damping involved. As you will see, the matrix notation we have been using to write and solve the EOMs for coupled linear oscillators can be readily extended to a damped system.



The two carts in the figure above have equal masses m. They are joined by identical but separate springs of force constant k to separate walls. Cart 2 rides in cart 1 as shows, and cart 1 is filled with molasses, whose viscous drag supplies the coupling between the two carts. The drag force has magnitude βmv where v is the relative velocity of the two carts.

a) Write down the equations of motion of the two carts using as coordinates x_1 and x_2, the displacements of the carts to the right of their equilibrium positions. Show that the EOM can be written in matrix form as

$$\mathbf{I}\ddot{x} + \beta\mathbf{D}\dot{x} + \omega_0^2\mathbf{I}x = 0$$

b) The next step is to "guess the solution form". Let's try normal mode form, but with a slight variation - just like you did with SHOs. Assuming that the drag force is weak ($\beta < \omega_0$), show that you do get two solutions of this form with $\omega = i\omega_0$ or $\omega = \beta + i\sqrt{\omega_0^2 - \beta^2}$.

c) Describe the motions corresponding to each normal mode, using words or sketches. Also, explain physically why one of the modes is damped but the other is not.

a)

$$m\begin{pmatrix}\ddot{x_1}\\ \ddot{x_2}\end{pmatrix} = \begin{pmatrix}-kx_1 + \beta m(\dot{x_2} - \dot{x_1})\\ kx_2 - \beta m(\dot{x_2} - \dot{x_1})\end{pmatrix}$$

$$\implies \begin{pmatrix}\ddot{x_1}\\ \ddot{x_2}\end{pmatrix} = \begin{pmatrix}-\frac{k}{m}x_1 + \beta(\dot{x_2} - \dot{x_1})\\ \frac{k}{m}x_2 - \beta(\dot{x_2} - \dot{x_1})\end{pmatrix}$$

$$\implies \begin{pmatrix}\ddot{x_1}\\ \ddot{x_2}\end{pmatrix} = -\begin{pmatrix}\omega_0^2 & 0\\ 0 & \omega_0^2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} - \beta\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot{x_1}\\ \dot{x_2}\end{pmatrix}$$

$$\implies \begin{pmatrix}\ddot{x_1}\\ \ddot{x_2}\end{pmatrix} + \beta\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot{x_1}\\ \dot{x_2}\end{pmatrix} + \begin{pmatrix}\omega_0^2 & 0\\ 0 & \omega_0^2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 0$$

b) We use

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}A_1\\ A_2\end{pmatrix}e^{\omega t}$$

Therefore,

$$[\omega^2\mathbf{I} + \omega\beta\mathbf{D} + \omega_0^2\mathbf{I}]x = 0$$

This needs to be diagonalized before anything else can be done.

$$[\omega^2\mathbf{I} + \omega\beta\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1} + \omega_0^2\mathbf{I}] = 0$$

where $\Lambda = \begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}$, $U = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$ and $U^{-1} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$ Therefore,

$$[\omega^2\mathbf{U}^{-1}\mathbf{U} + \omega\beta\mathbf{\Lambda} + \omega_0^2\mathbf{U}^{-1}\mathbf{U}] = 0$$

$$\implies [\omega^2\mathbf{I} + \omega\beta\mathbf{\Lambda} + \omega_0^2\mathbf{I}] = 0$$

Therefore,

$$\omega^2 + \omega_0^2 = 0, \quad \omega^2 + 2\beta\omega + \omega_0^2 = 0$$

and

$$\omega = i\omega_0, \quad \omega = -\beta + i\sqrt{\omega_0^2 - \beta^2}$$

c) Corresponding eigenvectors are

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 1\end{pmatrix}$$

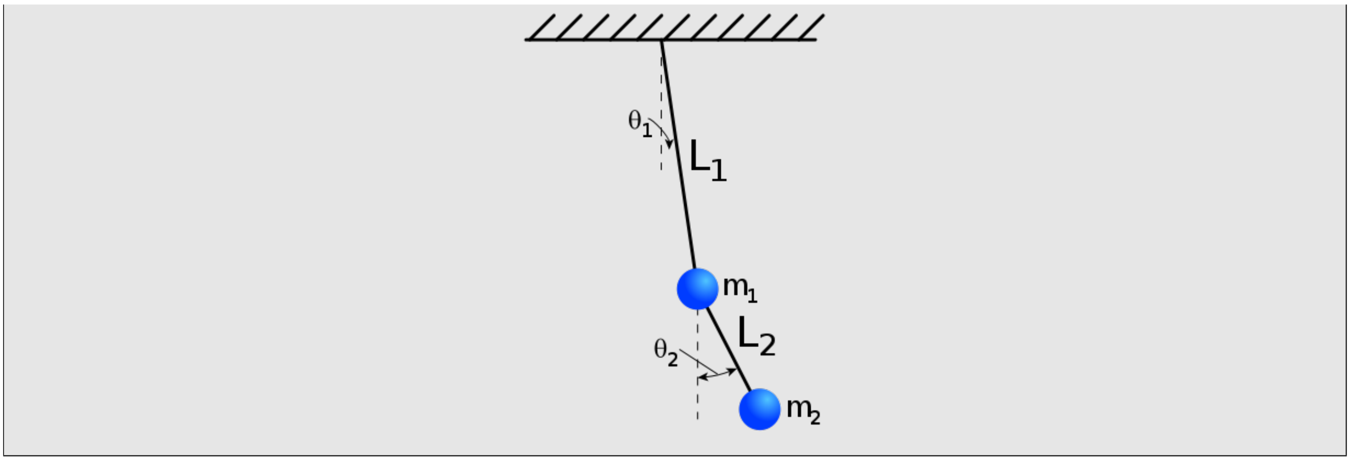
The first eigenvector corresponds to the case when motion is symmetric and damping never comes into play i.e. there is no relative velocity. In the antisymmetric case there is relative motion and damping is present.

Problem 3

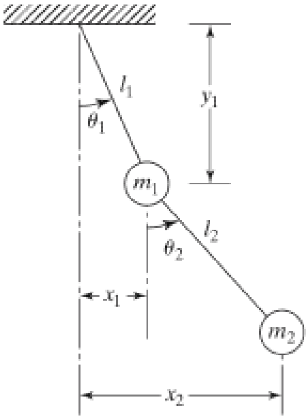
Find the normal modes for a double pendulum. Use the equation for Kinetic and Potential energy and use the following formulas:

$$\mathbf{M}\ddot{q} = -Kq, \quad \mathbf{M}_{ij} = \frac{\partial^2 T}{\partial\dot{q}_i\partial\dot{q}_j}|_{q=0} \quad \& \quad \mathbf{K}_{ij} = \frac{\partial^2 U}{\partial q_i\partial q_j}|_{q=0}$$

When calculating the value of the eigenfrequencies, you can assume $m_1 = m_2 = m$ and $l_1 = l_2 = l$.



Solution:



We obtain the equations

$$
\begin{aligned}
x_1 &= l_1 \sin\theta_1 \\
y_1 &= -l_1 \cos\theta_1 \\
x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 \\
y_2 &= -l_1 \cos\theta_1 - l_2 \cos\theta_2
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
\dot{x_1} &= l_1 \cos\theta_1 \dot{\theta_1} \\
\dot{y_1} &= l_1 \sin\theta_1 \dot{\theta_1} \\
\dot{x_2} &= l_1 \cos\theta_1 \dot{\theta_1} + l_2 \cos\theta_2 \dot{\theta_2} \\
\dot{y_2} &= l_1 \sin\theta_1 \dot{\theta_1} + l_2 \sin\theta_2 \dot{\theta_2}
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
\dot{x_1}^2 + \dot{y_1}^2 &= l_1^2 \dot{\theta_1}^2 \\
\dot{x_2}^2 + \dot{y_2}^2 &= l_1^2 \dot{\theta_1}^2 + l_2^2 \dot{\theta_2}^2 + 2 l_1 l_2 \cos(\theta_2 - \theta_1) \dot{\theta_1} \dot{\theta_2}
\end{aligned}
$$

Therefore,

$$
T = \frac{1}{2} m_1 (\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2} m_2 (\dot{x_2}^2 + \dot{y_2}^2) = \frac{m_1}{2} (l_1^2 \dot{\theta_1}^2) + \frac{m_2}{2} (l_1^2 \dot{\theta_1}^2 + l_2^2 \dot{\theta_2}^2 + 2 l_1 l_2 \cos(\theta_2 - \theta_1) \dot{\theta_1} \dot{\theta_2})
$$

and

$$V = (m_1 + m_2)gl_1(1 - \cos\theta_1) + m_2gl_2(1 - \cos\theta_2)$$

Therefore,

$$M = \begin{pmatrix} m_1l_1^2 + m_2l_2^2 & m_2l_1l_2 \\ m_2l_1l_2 & m_2l_2^2 \end{pmatrix}, \quad K = \begin{pmatrix} (m_1 + m_2)gl_1 & 0 \\ 0 & m_2gl_2 \end{pmatrix}$$

Now, assuming

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t}$$

we get,

$$[\mathbf{M}\omega^2 - \mathbf{K}] \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0$$

Using

$$det(\mathbf{M}\omega^2 - \mathbf{K}) = 0$$

and $m_1 = m_2 = m$ and $l_1 = l_2 = l$.

$$\omega_{1,2} = \sqrt{\frac{g}{l}}\sqrt{2 \pm \sqrt{2}}$$

and non normalized eigenvectors are

$$\begin{pmatrix} 1 \\ \mp\sqrt{2} \end{pmatrix}$$