Res Session 2

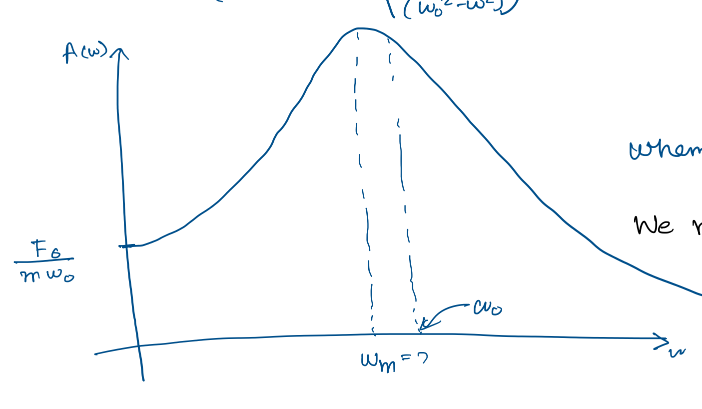
Monday, October 2, 2023 01:51

Last week we saw,

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t}$$

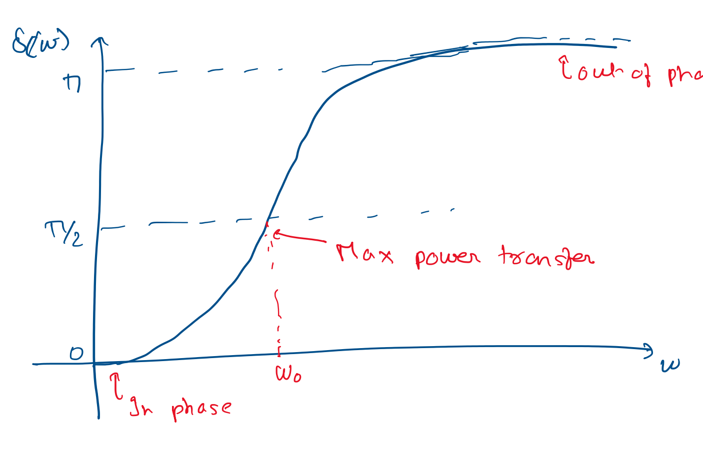
$$A(\omega) = \frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2\right]^{1/2}}$$

$$\delta(\omega) = \tan^{-1}\left(\frac{\gamma\omega}{(\omega_0^2-\omega^2)}\right)$$



When $\gamma = 0$, $\omega_m = \omega_0$ & $A_{max} \to \infty$

We nonetheless will refer to $\omega = \omega_0$ as resonance for all intents and purposes



$\omega \to 0$, $\delta \to 0^+$

$\omega \to \omega_0$, $\delta = \pi/2$

$\omega \to \infty$, $\delta \to \pi$

$\tan(\delta) = 0^-$

All that pain to write,

$$x(t) = \underbrace{\frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right]^{1/2}} \cos(\omega t - \delta)}_{} + x_{damped}(t)$$

But where are my unknown constants

(not required for steady state solns).

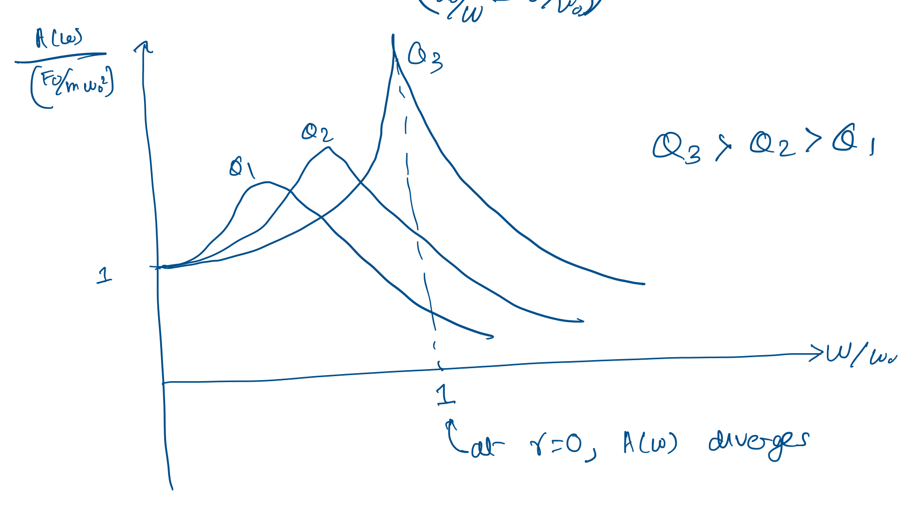
Mexico city 1985 $\to T = \frac{2\pi}{\omega_0} \sim 0.1$ # of floors $\leftarrow \omega_0$ smaller for more floors.

$\leftarrow$ soil attenuates high freqs.

As always, $Q = \omega_0/\gamma$

$$\& \; A(\omega) = \frac{(F_0/m)(\omega_0/\omega)}{\left[(\omega_0/\omega - \omega/\omega_0)^2 + 1/Q^2\right]^{1/2}}$$

$$\delta(\omega) = \tan^{-1}\left(\frac{1/Q}{\omega_0/\omega - \omega/\omega_0}\right)$$



$Q_3 > Q_2 > Q_1$

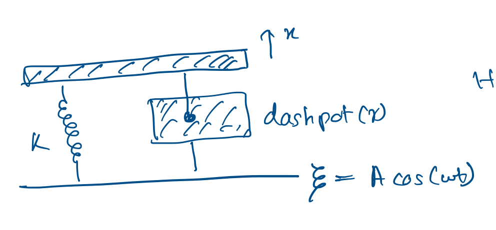
In all other cases,

$$\left.\frac{d(A(\omega))}{d\omega}\right|_{\omega_m} = 0 \quad \& \quad \left.\frac{d^2A(\omega)}{d\omega^2}\right|_{\omega_m} < 0$$

$$\Rightarrow \therefore \omega_m = \omega_0\left(1 - \frac{1}{2Q^2}\right)^{1/2}$$

Lets solve one quick problem

① Seismic isolation stacks



How do i save my experiment?

Just write eqns of motion like usual.

$$m\ddot{x} = -k(x-\xi) - b\frac{d}{dt}(x-\xi) - mg$$

$$\Downarrow$$

$$m\ddot{x} = -k(x-\xi) - b\frac{d}{dt}(x-\xi)$$

We take $X = x - \xi \Rightarrow x = X + \xi$

$$m\ddot{x} = m\ddot{X} + m\ddot{\xi} = -kX - b\dot{X}$$

$$\Rightarrow m\ddot{X} + b\dot{X} + kX = -m\ddot{\xi}$$

eq. to

$$\underbrace{\ddot{X} + \gamma\dot{X} + \omega_0^2 X = \omega^2 A e^{i\omega t}}_{}$$

Yay, we know how to solve this.

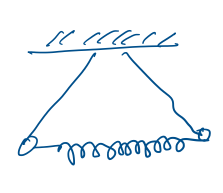
I am not solving the scenarios for complex impedances here as there is not much to add from what you have learnt in class. But we can always discuss your doubts after we are done today.

Lets look at coupled oscillators instead

$\hookrightarrow$ the world afterall is not just one oscillator.

We use it to study interacting oscillators.

Lets look at a simple system first.



N interacting particles

$$\left.\begin{aligned} \ddot{x}_1 &= f_1(x_1, x_2, \ldots, x_N) \\ \ddot{x}_2 &= f_2(x_1, x_2, \ldots, x_N) \\ &\vdots \\ \ddot{x}_N &= f_N(x_1, x_2, \ldots x_N) \end{aligned}\right\} \text{force is linear for small oscillations}$$

$$\Downarrow$$

I can find linear combinations

linear combination of x_i's $\left\{\begin{aligned} &u_1(x_1, \ldots, x_N) \\ &u_2(x_1, \ldots, x_N) \\ &\vdots \\ &u_N(x_1, \ldots, x_N) \end{aligned}\right.$ s.t. $\left.\begin{aligned} \ddot{u}_1 &= -\omega_1^2 u_1 \\ \ddot{u}_2 &= -\omega_2^2 u_2 \\ &\vdots \\ \ddot{u}_N &= -\omega_N^2 u_N \end{aligned}\right\}$

Normal coordinates

set of N decoupled oscillators

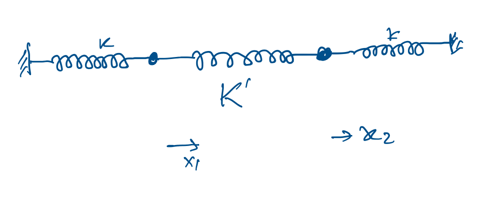
N d.o.f. $\to$ N normal coordinates

So what are normal modes?

Special configurations of N particles s.t. each particle of this system oscillates at same frequency.

There are N normal modes.

② Lets look at the following system



Assume $x_1 > x_2$

$$m\ddot{x}_1 = -kx_1 - k'(x_1 - x_2) = -(k+k')x_1 + k'x_2$$

$$m\ddot{x}_2 = -kx_2 + k'(x_1 - x_2) = k'x_1 - (k+k')x_2$$

$$\therefore \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -(k+k')/m & k'/m \\ k'/m & -(k+k')/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\ddot{X} = MX$$

$\therefore$ Normal modes $\to$ Ansatz $\to$

$$x_1(t) = A_1 e^{i(\omega t + \phi_1)} = \tilde{A}_1 e^{i\omega t}$$

$$x_2(t) = A_2 e^{i(\omega t + \phi_2)} = \tilde{A}_2 e^{i\omega t}$$

$$\vdots$$

$$x_N(t) = A_N e^{i(\omega t + \phi_N)} = \tilde{A}_N e^{i\omega t}$$

$$\begin{pmatrix} -\omega^2 x_1 \\ -\omega^2 x_2 \end{pmatrix} = -M\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow -\omega^2 e^{i\omega t}\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} = -M\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} e^{i\omega t}$$

$$\Rightarrow M\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} = \underbrace{\omega^2}_{\text{eigenvalues}} \overbrace{\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix}}^{\text{eigenvectors}}$$

$\therefore$ For eigenvalues,

$$\det(M - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} \frac{k+k'}{m} - \lambda & -k'/m \\ -k'/m & \frac{k+k'}{m} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{k+k'}{m} - \lambda\right)^2 - \left(\frac{k'}{m}\right)^2 = 0$$

$$\therefore \lambda = \omega^2 = k/m \;\&\; k+2k'/m$$

$A_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ oscillating in phase

$A_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ oscillating out of phase

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_1 t + \phi_1) + A_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_2 t + \phi_2)$$

eigenvectors span the space.

$$\therefore x_1(t) = A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2)$$

$$x_2(t) = A_1\cos(\omega_1 t + \phi_1) - A_2\cos(\omega_2 t + \phi_2)$$

What about the normal coordinates,

$$u_1 = x_1 + x_2 = 2A_1\cos(\omega_1 t + \phi_1)$$

$$u_2 = x_1 - x_2 = 2A_2\cos(\omega_2 t + \phi_2)$$

as expected.

Alternately,

$$\omega_1 \colon \begin{pmatrix} 3 \\ 2 \end{pmatrix} A_1, \quad \omega_2 \colon \begin{pmatrix} 1 \\ -5 \end{pmatrix} A_2$$

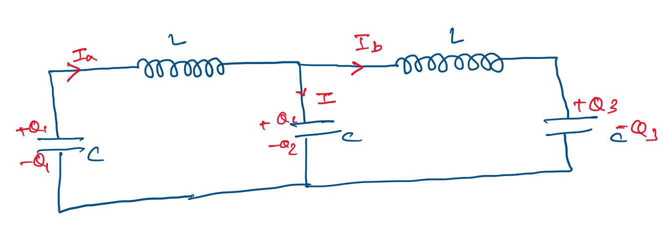
$$\therefore x_1(t) = 3A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2)$$

$$x_2(t) = 2A_1\cos(\omega_1 t + \phi_2) - 5A_2\cos(\omega_2 t + \phi_2)$$

$$\therefore u_1(t) = 5x_1 + x_2$$

$$u_2(t) = 2x_1 - 3x_2$$

③



Find the modes?

How many?

$$-L\dot{I}_a - \frac{Q_2}{C} + \frac{Q_1}{C} = 0 \quad \Rightarrow \quad LC\dot{I}_a = Q_1 - Q_2$$

$$-L\dot{I}_b - \frac{Q_3}{C} + \frac{Q_2}{C} = 0 \qquad\quad LC\dot{I}_b = Q_2 - Q_3$$

$$\& \; I = I_a - I_b$$

$$\Rightarrow LC\ddot{I}_a = -I_a - I \qquad \dot{Q}_1 = -I_a$$

$$LC\ddot{I}_b = -I - I_b \qquad \dot{Q}_2 = I$$

$$\dot{Q}_3 = I_b$$

$$\Rightarrow LC\ddot{I}_a = -2I_a + I_b$$

$$LC\ddot{I}_b = I_a - 2I_b$$

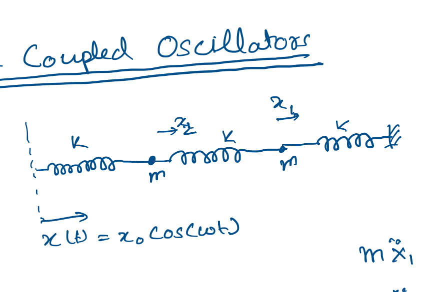
$$\begin{matrix} \ddot{I}_a \\ \ddot{I}_b \end{matrix} = \begin{pmatrix} -2/LC & 1/LC \\ 1/LC & -2/LC \end{pmatrix}\begin{pmatrix} I_a \\ I_b \end{pmatrix} \Rightarrow \begin{aligned} \omega_1^2 &= 3/LC \to \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \omega_2^2 &= 1/LC \to \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

$$I_a(t) = A\cos(\omega_1 t + \phi_1) + B\cos(\omega_2 t + \phi_2)$$

$$I_b(t) = A\cos(\omega_1 t + \phi_1) - B\cos(\omega_2 t + \phi_2)$$

④ Forced Coupled Oscillators

$x > x_2 > x_1$



$$m\ddot{x}_1 = k(x_2 - x_1) - kx_1$$

$$m\ddot{x}_2 = k(x - x_2) - k(x_2 - x_1)$$

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2 + kx_0\cos(\omega t)$$

$$\Rightarrow \left.\begin{aligned} \ddot{x}_1 - \frac{2k}{m}x_1 - \frac{k}{m}x_2 &= 0 \\ \ddot{x}_2 - \frac{k}{m}x_1 + \frac{2k}{m}x_2 &= \frac{kx_0}{m}\cos\omega t \end{aligned}\right\} \text{cant write as } \ddot{X} = -MX$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 2k/m & -k/m \\ -k/m & 2k/m \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{kx_0}{m}\cos\omega t \end{pmatrix}$$

What are the normal modes in the steady state?

—each particle matches the driving frequency.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} e^{i\omega t}$$

$$\Rightarrow -\omega^2\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} + \begin{pmatrix} 2k/m & -k/m \\ -k/m & 2k/m \end{pmatrix}\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ kx_0/m \end{pmatrix}$$

$$\Rightarrow \underbrace{\begin{pmatrix} 2k/m - \omega^2 & -k/m \\ -k/m & 2k/m - \omega^2 \end{pmatrix}}_{Y}\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ kx_0/m \end{pmatrix}$$

If this is zero, $\det Y = 0$ (Y is singular)

$$\text{or} \quad \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} = Y^{-1}\begin{pmatrix} 0 \\ kx_0/m \end{pmatrix}$$

$$\tilde{A}_1 = \frac{1}{(3k/m - \omega^2)(k/m - \omega^2)} \frac{k^2x_0}{m}$$

$$\tilde{A}_2 = \frac{(2k/m - \omega^2)}{(3k/m - \omega^2)(k/m - \omega^2)} \frac{kx_0}{m}$$