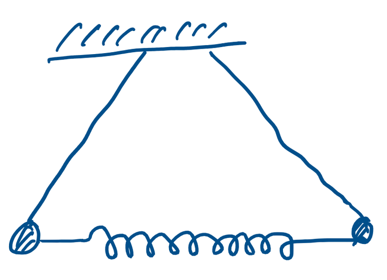
Lecture 7

Wednesday, October 19, 2022 12:35 AM

Coupled Oscillators — because the world is not one oscillator (mostly!)

↳ We use it to study interacting oscillators.

We will deal with simple systems first.



Lets talk about whats happening first.

N-interacting particles

$$\left.\begin{aligned} \ddot{x}_1 &= f_1(x_1, x_2, \dots, x_N) \\ \ddot{x}_2 &= f_2(x_1, x_2, \dots, x_N) \\ &\vdots \\ \ddot{x}_N &= f_N(x_1, x_2, \dots, x_N) \end{aligned}\right\}$$

for small oscillations my force laws are linear. $(-Kx_i)$

$\Downarrow$ I can then find

Linear combinations of x_is

$$\left\{\begin{aligned} &u_1(x_1, \dots, x_N) \\ &u_2(x_1, \dots, x_N) \\ &\vdots \\ &u_N(x_1, \dots, x_N) \end{aligned}\right. \quad \text{s.t.} \quad \begin{aligned} \ddot{u}_1 &= -\omega_1^2 u_1 \\ \ddot{u}_2 &= -\omega_2^2 u_2 \\ &\vdots \\ \ddot{u}_N &= -\omega_N^2 u_N \end{aligned}$$

Normal Coordinates (this is what we find)

Set of N decoupled Oscillators.

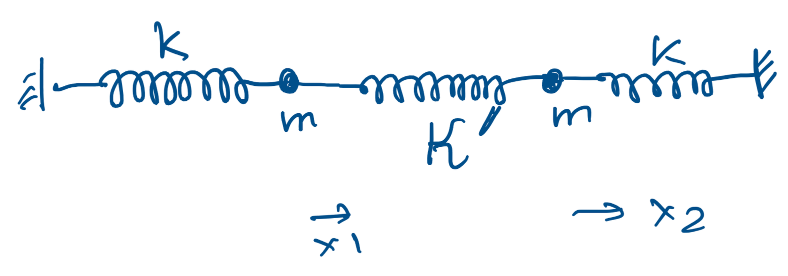
N d.o.f → N normal coordinates

So what are normal modes?

Special configurations of N particles s.t. each particle of this system oscillates at same frequency.

There are N normal modes.

So let's look at some examples,



Assume $x_1 > x_2$

$$m\ddot{x}_1 = -Kx_1 - K'(x_1 - x_2) = -(K+K')x_1 + K'x_2$$
$$m\ddot{x}_2 = -Kx_2 + K'(x_1 - x_2) = K'x_1 - (K+K')x_2$$

yay! linear eqs

$$\therefore \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{(K+K')}{m} & \frac{K'}{m} \\ \frac{K'}{m} & -\frac{(K+K')}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\ddot{X} = -MX$$

Normal Modes ⟶ Ansatz: → $x_1(t) = A_1 e^{i(\omega t + \phi_1)} = \tilde{A}_1 e^{i\omega t}$

$x_2(t) = A_2 e^{i(\omega t + \phi_2)} = \tilde{A}_2 e^{i\omega t}$

$\vdots$

$x_N(t) = A_N e^{i(\omega t + \phi_N)} = \tilde{A}_N e^{i\omega t}$

$$\begin{pmatrix} -\omega^2 x_1 \\ -\omega^2 x_2 \end{pmatrix} = -M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow -\omega^2 e^{i\omega t} \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} = -M \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} e^{i\omega t}$$

$$\Rightarrow M \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} = \omega^2 \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix}$$

eigenvectors

eigenvalues

For eigenvalues,

$$\det(M - \lambda I) = 0$$

$$\begin{vmatrix} \frac{K+K'}{m} - \lambda & -K'/m \\ -K'/m & \frac{K+K'}{m} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{K+K'}{m} - \lambda\right)^2 - \frac{K'^2}{m^2} = 0$$

$$\Rightarrow \lambda = \omega^2 = \sqrt{K/m}\ ,\ \sqrt{\frac{K+2K'}{m}}$$

$A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ → oscillating in phase

$A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ → oscillation out of phase

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t + \phi_1) \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t + \phi_2)$$

(need two initial condns) (need two initial condns)

All motions are linear combinations of eigenmodes

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t + \phi_1) + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t + \phi_2)$$

(eigenvectors span space)

$$\therefore x_1(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$$
$$x_2(t) = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2)$$

So what are my normal coordinates?

get rid of ω_2

$$\left.\begin{aligned} u_1 &= x_1 + x_2 = 2A_1 \cos(\omega_1 t + \phi_1) \\ u_2 &= x_1 - x_2 = 2A_2 \cos(\omega_2 t + \phi_2) \end{aligned}\right\}$$

Not because $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Lets say

$$\omega_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} A_1 \ ; \ \omega_2 = \begin{pmatrix} 1 \\ -5 \end{pmatrix} A_2$$

Normal coordinates?

$$x_1(t) = 3A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$$
$$x_2(t) = 2A_1 \cos(\omega_2 t + \phi_1) - 5A_2 \cos(\omega_2 t + \phi_2)$$

$$\therefore u_1(t) = 5x_1 + x_2$$
$$u_2(t) = 2x_1 - 3x_2$$