Lecture 6

Thursday, October 13, 2022 10:56 PM

Quick Review

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t}$$

$$x(t) = \mathrm{Re}\,(z(t)) = \begin{cases} \overbrace{c_1 e^{-\gamma t/2} \cos(\omega_d t + \phi)}^{\text{complementary}} + \overbrace{A(\omega)\cos(\omega t - \delta)}^{\text{particular}}, & \gamma < 2\omega_0 \text{ (lightly damped)} \\ (c_1 + c_2 t)\, e^{-\gamma t/2} + A(\omega)\cos(\omega t - \delta), & \gamma = 2\omega_0 \text{ (critical damping)} \\ \underbrace{c_1 e^{-\alpha_1 t} + c_2 e^{-\alpha_2 t}}_{\text{Depends on initial conditions (die out)}} + A(\omega)\cos(\omega t - \delta), & \gamma > 2\omega_0 \text{ (overdamped)} \end{cases}$$

$$A(\omega) = \frac{F_0/m}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}}$$

$$\delta(\omega) = \tan^{-1}\left(\frac{\gamma\omega}{(\omega_0^2 - \omega^2)}\right)$$

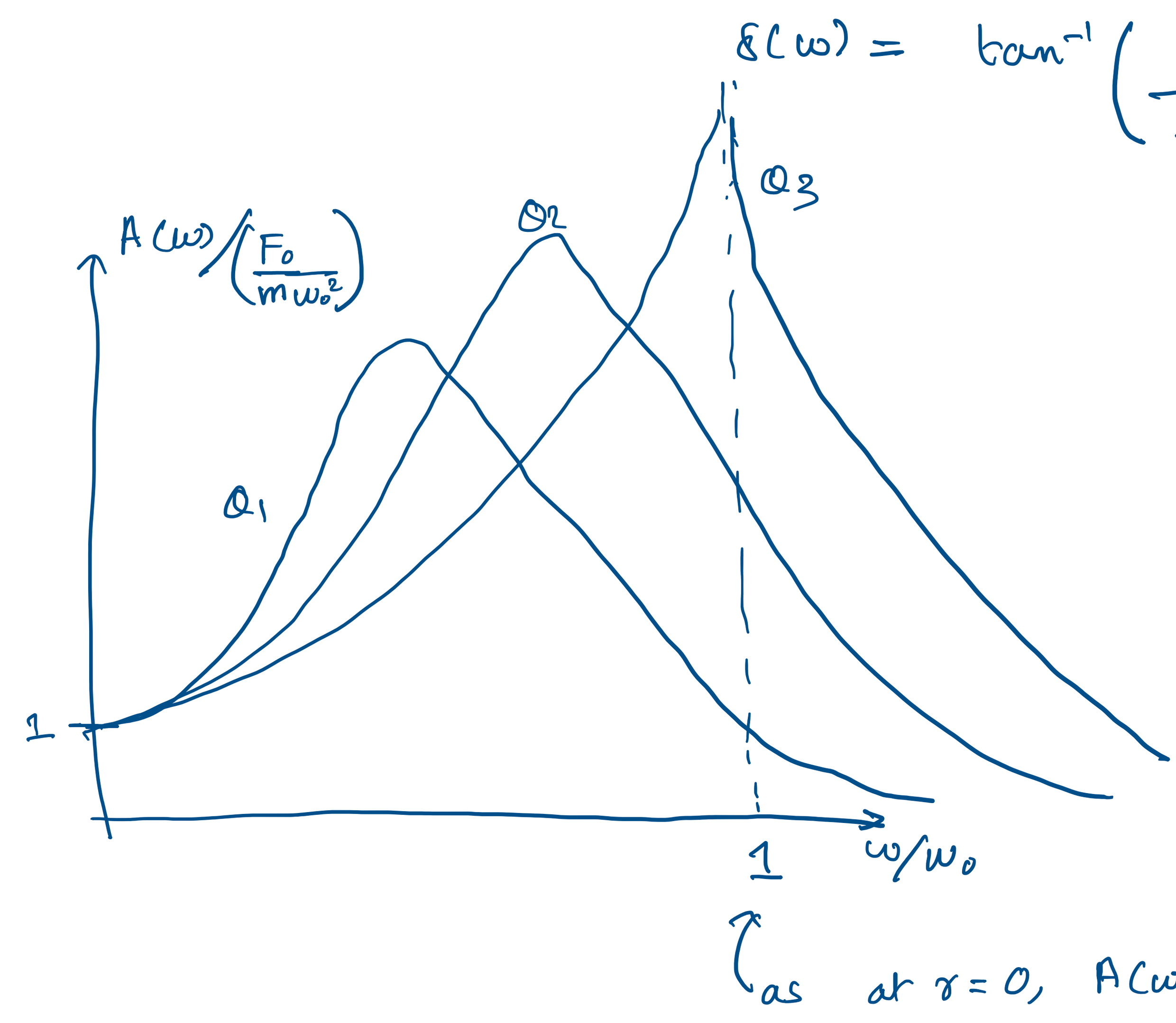
Just like before,

$$Q = \omega_0/\gamma$$

then

$$A(\omega) = \frac{F_0}{m} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

$$\delta(\omega) = \tan^{-1}\left(\frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}}\right)$$



$Q_3 > Q_2 > Q_1$

as at $\gamma = 0$, $A(\omega)$ diverges

But where does it peak for the generic case?

Just differentiate

For ω_m,

$$\left.\frac{d A(\omega)}{d\omega}\right|_{\omega_m} = 0$$

$$\left.\frac{d^2 A(\omega)}{d\omega^2}\right|_{\omega_m} < 0$$

$$\omega_m = \omega_0 \left(1 - \frac{1}{2Q^2}\right)^{1/2}$$

$\gamma = 0$

$Q \to \infty$

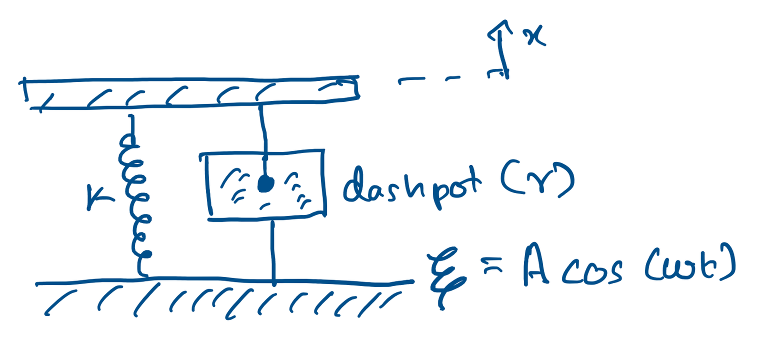
& $\omega_m = \omega_0$

Lets solve some problems now,

① LIGO

(we will use a different setup but the concept is similar)

Let us take seismic isolation stacks



How do I remove this motion from my table top?

Describe the motion of my table top

$$m\ddot{x} = -k(x - \xi) - \gamma \frac{d}{dt}(x - \xi) - mg$$

$$\Downarrow$$

$$m\ddot{x} = -k(x - \xi) - \gamma \frac{d}{dt}(x - \xi)$$

We take $X = x - \xi \Rightarrow x = X + \xi$

$$m\ddot{x} = m\ddot{X} + m\ddot{\xi} = -kX - b\dot{X}$$

$$\Rightarrow m\ddot{X} + b\dot{X} + kX = -m\ddot{\xi}$$

$$\Rightarrow \underbrace{\ddot{X} + \gamma\dot{X} + \omega_0^2 X = \omega^2 A\cos(\omega t)}_{\text{Yay!! we know how to solve this}}$$

Steady state

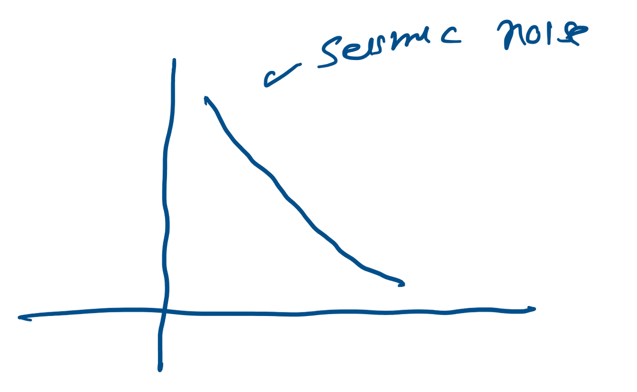
$$X(t) = C(\omega)\cos(\omega t - \delta(\omega))$$

$$C(\omega) = \frac{A\omega^2}{\left[(\omega_0^2 - \omega^2) + \gamma^2\omega^2\right]^{1/2}}$$

$$x(t) = C(\omega)\cos(\omega t - \delta(\omega)) + A\cos(\omega t)$$

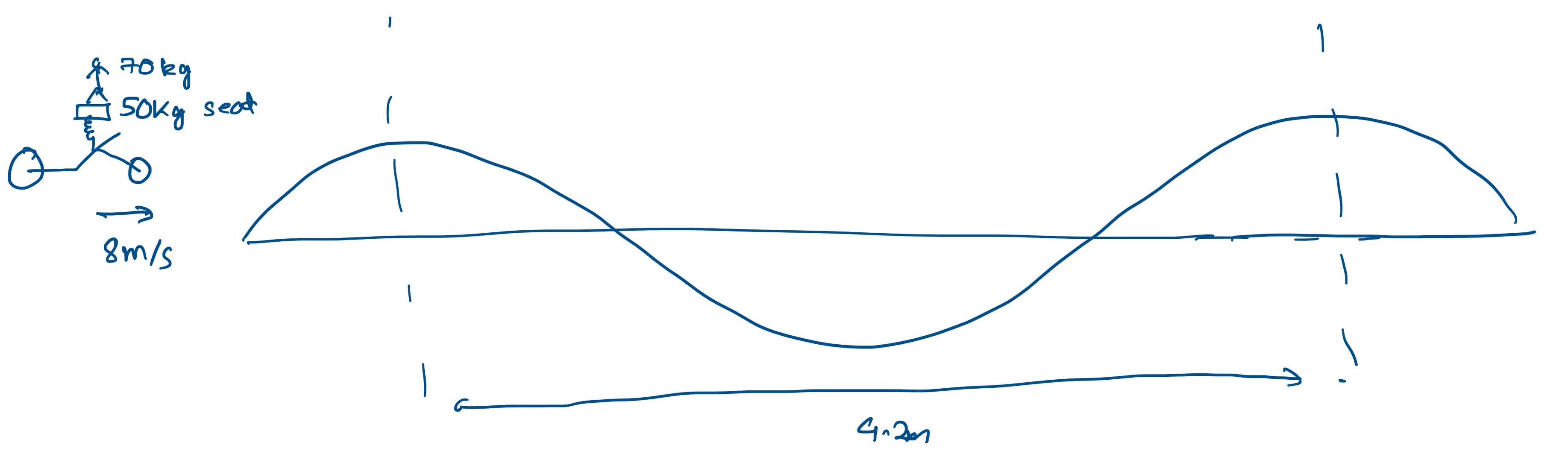
$$\max(x(t)) \begin{cases} = A, & \omega^2 = 2\omega_0^2 \\ > A, & \omega^2 < 2\omega_0^2 \\ < A, & \omega^2 > 2\omega_0^2 \end{cases}$$

Amplitude



LIGO actually uses setups like quadruple pendulum to acheive damping,

② Bumps on a road.



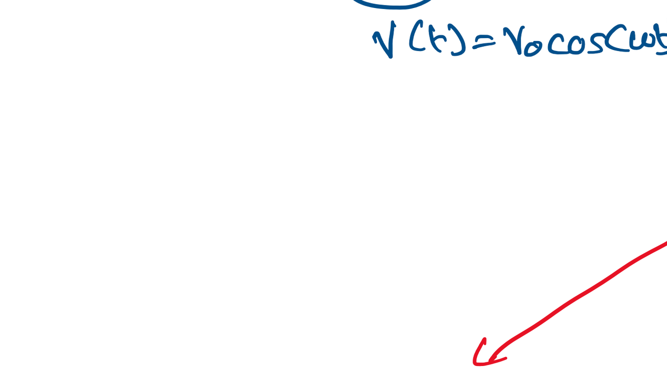
Is there a need to call 911?

— there can be resonance

$$\Delta t = 4.2/8 \sim 1/2 \Rightarrow f = 2\,Hz$$

So if for some k, $f_0 \sim 2\,Hz$ we better get help.

③



Find the behaviour of the circuit?

$\rightarrow$ $I_1 + I_2 = I$

$\rightarrow$ $V_a - L\frac{dI_1}{dt} - I_1 R + V(t) = V_a$

$\rightarrow$ $-\frac{q(t)}{C} + V(t) = 0$

& $I_2 = \dot{q} = -V_0 C\omega \sin(\omega t)$

$$\dot{I}_1 + \frac{R}{L} I_1 = \frac{V_0}{L}\cos(\omega t)$$

$$\dot{I}_1 + \frac{R}{L} I_1 = 0$$

$$I_1 = e^{-R/L\, t + C} = A e^{-R/L\, t}$$

(You can also use $e^{\alpha t}$ as an ansatz)

$$\dot{I}_1 + R/L\, I_1 = \frac{V_0}{L}\cos(\omega t)$$

$$\dot{\tilde{I}}_1 + \frac{R}{L}\tilde{I}_1 = \frac{V_0}{L} e^{i\omega t}$$

Guess $\tilde{I}_1 = A e^{i(\omega t - \delta)}$

$$A e^{-i\delta}\left[i\omega + \frac{R}{L}\right] = \frac{V_0}{L}$$

$$\Rightarrow A\left[i\omega + \frac{R}{L}\right] = \frac{V_0}{L} e^{i\delta}$$

$$\Rightarrow A = \frac{V_0/L}{\left[(R/L)^2 + \omega^2\right]}$$

$$\& \quad \delta = \tan^{-1}\left(\frac{\omega L}{R}\right)$$