Lecture 5

Wednesday, October 12, 2022 8:49 AM

Driven damped harmonic oscillator

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

driving freq

} → Why just this

→ generalized force

natural freq

Two behaviors
→ ① Transient → damping and natural frequencies all die out
② Steady state ($t \to \infty$) → Only driving forces remain.

But let's look at some diff eqs. first

Inhomogenous Diff Equation

$$\ddot{y} + p(t)\dot{y} + q(t)y = g(t)$$ ← most general 2nd order ODE

Maths tells me I can solve it in two parts

Homogenous

$\ddot{y} + p(t)\dot{y} + q(t)y = 0$

Say our solution is

$y_c(t)$

↳ complimentary solution

(we saw this already)

Inhomogenous

$y_p(t)$ solution

↳ particular

(we will see this as we go ahead)

The claim is

$y(t) = y_c(t) + y_p(t)$ solve the inhomogenous eqn too by linearity.

So how do find $y_p(t)$ → Guess

e.g. $\ddot{y} - 4\dot{y} - 12y = 3e^{st}$

guess $y_p(t) = Ae^{st}$

$$25Ae^{st} - 20Ae^{st} - 12Ae^{st} = 3e^{st}$$

$$\Rightarrow A = -3/7$$

e.g. $\ddot{y} - 4\dot{y} - 12y = 2t^3 - t + 3$

guess $y_p(t) = At^3 + Bt^2 + Ct + D$ ← for nth order driving polynomial guess a similar polynomial

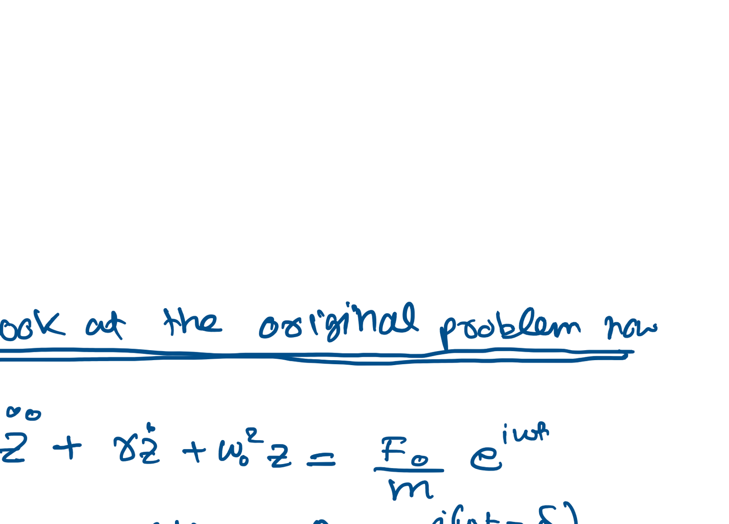
Plug & now match order by order

e.g. If $g(t) = 5\cos\omega t$

$y_p(t) = A\cos(\omega t) + B\sin(\omega t)$

This is called method of undetermined coefficients.

Quick review of complex no's



$z = Re^{i\phi}$

$= R\cos\phi + i\,R\sin\phi$

Lets look at the original problem now

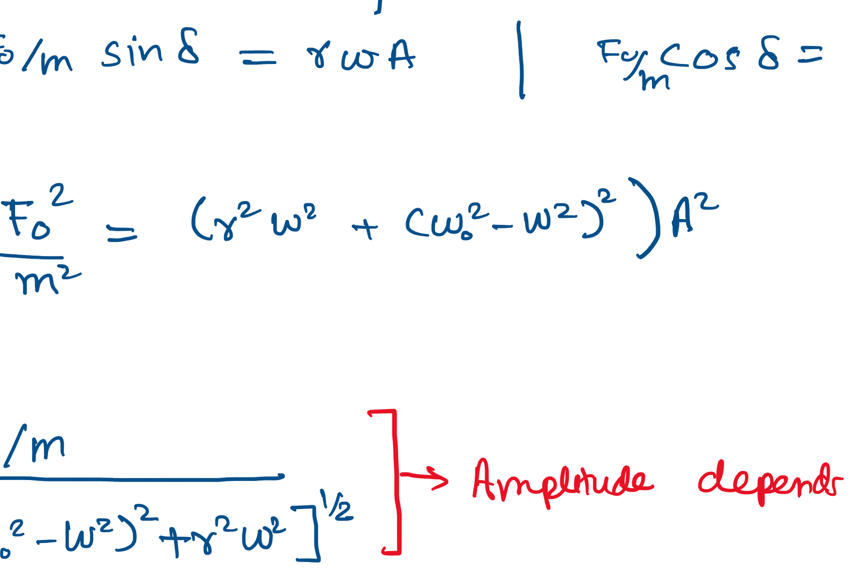
$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t}$$

Ansatz: ($t \to \infty$) $z_p(t) = Ae^{i(\omega t - \delta)}$

↑ phase angle by which driving force leads the displacement.

$$A\cancel{e^{i(\omega t-\delta)}}\left[-\omega^2 + i\gamma\omega + \omega_0^2\right] = \frac{F_0}{m}\cancel{e^{i\omega t}}$$

$$(\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m}e^{i\delta}$$

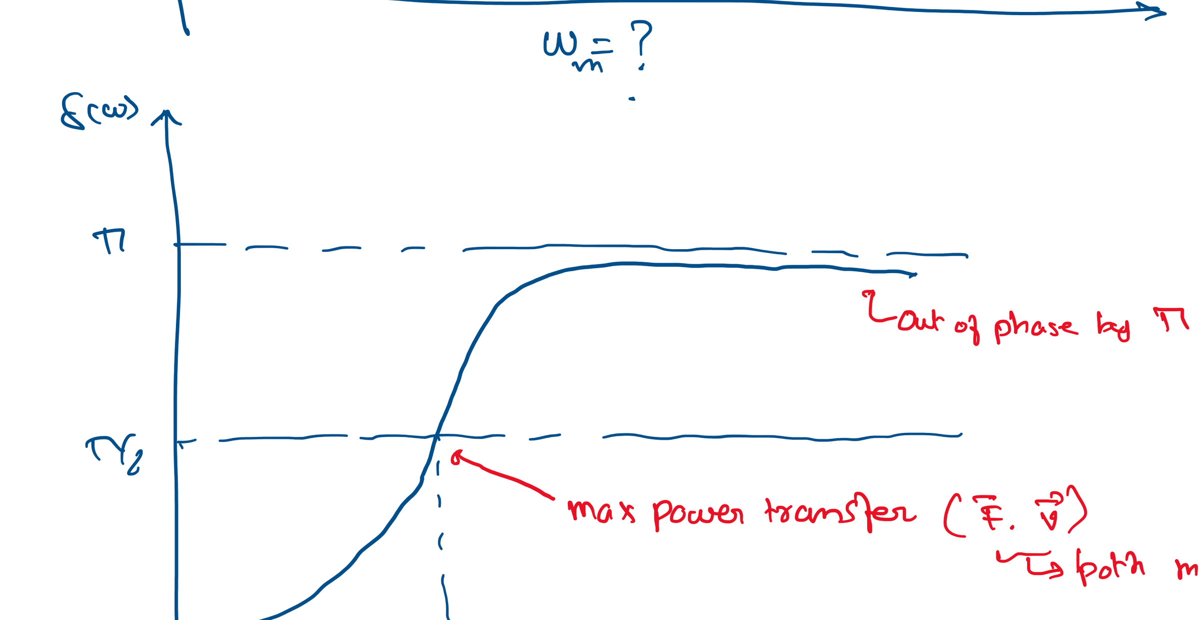


So $F_0/m \sin\delta = \gamma\omega A$ | $F_0/m\cos\delta = (\omega_0^2 - \omega^2)A$

$$\therefore \frac{F_0^2}{m^2} = \left(\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2\right)A^2$$

$$A(\omega) \equiv \frac{F_0/m}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}}$$ → Amplitude depends on frequency.

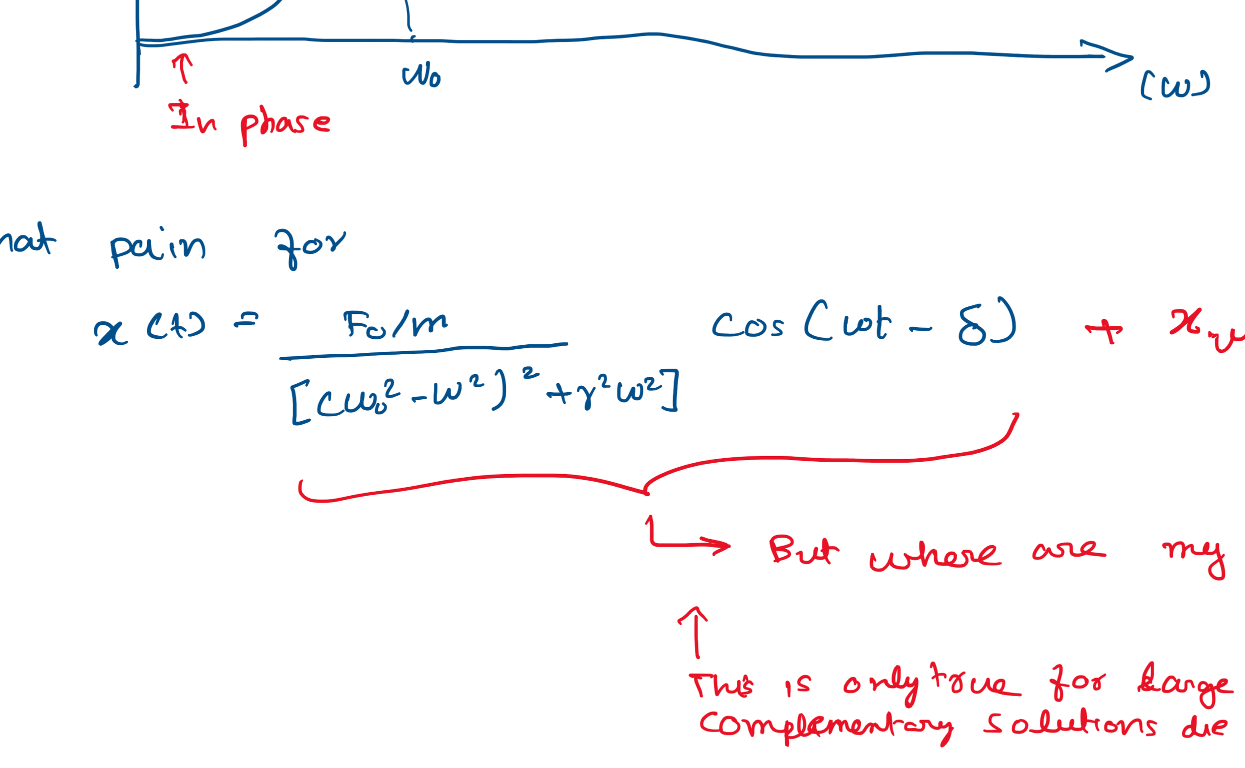
$$\delta = \tan^{-1}\left(\frac{\gamma\omega}{(\omega_0^2 - \omega^2)}\right)$$



← When $\gamma = 0$, $\omega_m = \omega_0$

In this case $A_{max} \to \infty$

We refer to it as resonance even if γ is non zero.



$\omega \to 0$, $\delta \to 0^+$

$\omega \to \omega_0$, $\delta = \pi/2$

$\omega \to \infty$, $\delta \to \pi$

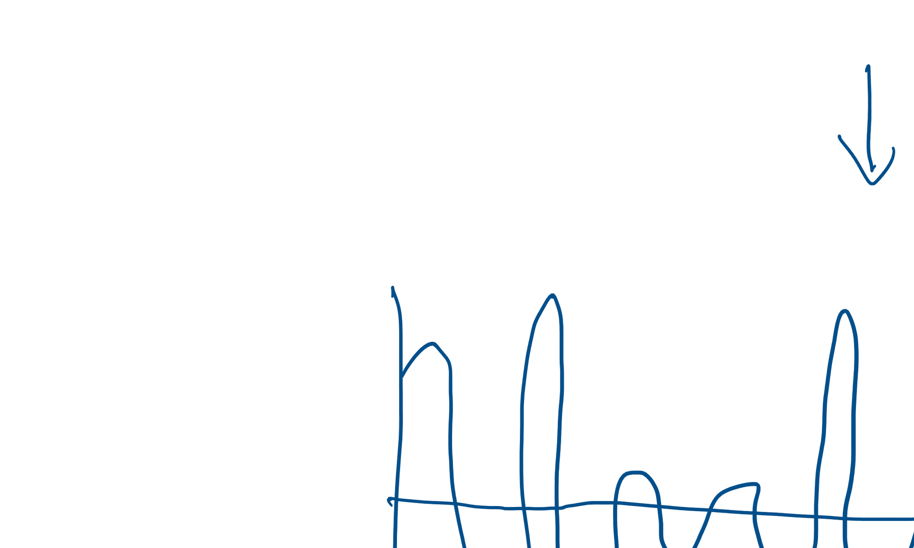
$\tan\delta \to 0^-$

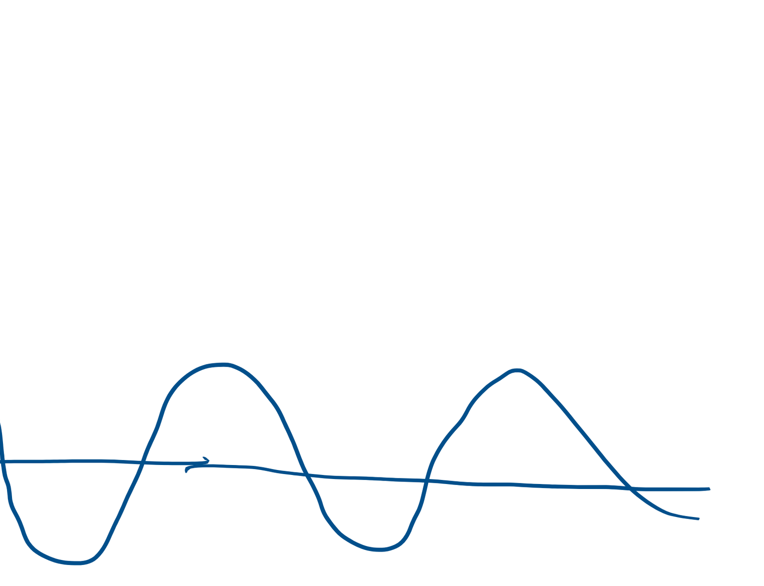
All that pain for

$$x(t) = \underbrace{\frac{F_0/m}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]}\cos(\omega t - \delta)} + x_{underdamped}(t)$$

→ But where are my unknown constants?

↑ This is only true for large time periods after complementary solutions die out





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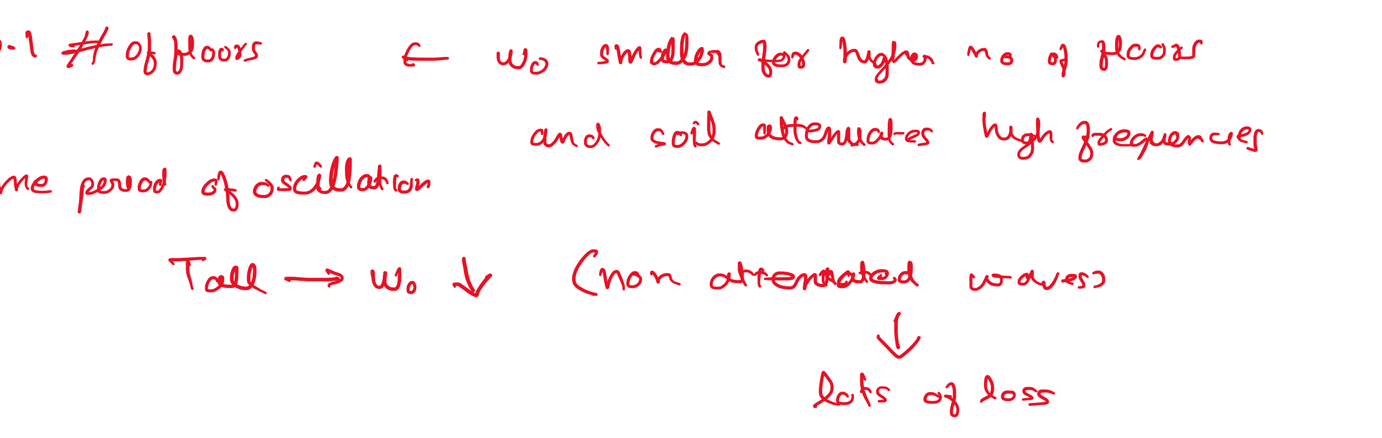

Mexico City 1985

$T = \frac{2\pi}{\omega_0} \sim 0.1$ # of floors ← ω_0 smaller for higher no of floors and soil attenuates high frequencies

↳ natural time period of oscillation

Tall → ω_0 ↓ (non attenuated waves)

↓

lots of loss

Tacoma Narrows