Lecture 4

Thursday, October 6, 2022 11:43 PM

Quick Recap

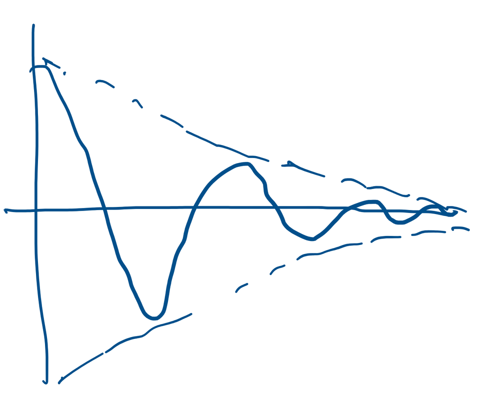
$$\ddot{z} + \gamma\dot{z} + w_0^2 z = 0$$

Ansatz: $z(t) = Ce^{\alpha t} \Rightarrow \alpha = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 4w_0^2}$

① Lightly / Under damped ($\gamma < 2w_0$)

$$\alpha = -\frac{\gamma}{2} \pm i w_u \;;\; w_u = w_0\sqrt{1 - \frac{\gamma^2}{4w_0^2}}$$

$$\therefore x(t) = Ae^{-\gamma t/2}\cos(w_u t + \phi)$$



② Overdamp ($\gamma > 2w_0$)

$$x(t) = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t}$$

$$\alpha_{1/2} = \gamma/2 \pm \frac{1}{2}\sqrt{\gamma^2 - 4w_0^2}$$

$$\alpha_1 > \alpha_2 > 0$$

} Never Oscillate

③ Critical damping ($\gamma = w_0$)

$$x(t) = (C_1 + C_2 t)e^{-\gamma t/2} = (C_1 + C_2 t)e^{-w_0 t}$$

no oscillations

[fastest return to equilibrium]

e.g. door or car shock absorbers

④ Very lightly damped.

$$\gamma \ll 2w_0$$

$$w_u = w_0\left(1 - \frac{1}{2}\frac{\gamma^2}{4w_0^2} + O\left(\frac{\gamma^4}{w_0^4}\right)\right)$$

$$\simeq w_0\left(1 - \frac{\gamma^2}{8w_0^2}\right)$$

$\therefore$ upto linear order

$$w_u \simeq w_0$$

As before, we will try to look at the energy picture too.

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$ } am i missing a friction / damping termed.

We just dont have conserved energy anymore.

$$\frac{dE}{dt} = \frac{1}{2}m(2\dot{x}\ddot{x}) + \frac{1}{2}k(2x\dot{x})$$

$$= \dot{x}(m\ddot{x} + kx)$$ [Recollect that for $\gamma = 0$, $m\ddot{x} + kx = 0$]

[But we do have $m\ddot{x} + b\dot{x} + kx = 0$]

$$= -b\dot{x}^2 \leq 0$$

loss of energy in system.

You can also see,

$$\frac{dE}{dt} = (-b\dot{x})\dot{x} = \vec{F}_{damp}\cdot\vec{v}$$ } Power dissipated by damping force

Lets look at more specific cases

Lightly damped

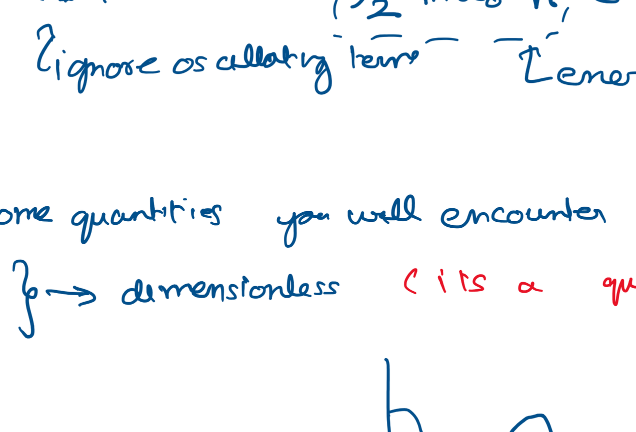
$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

using $x(t) = Ae^{-\gamma t/2}\cos(w_u t + \phi)$

$$E = \frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\cos(2w_u t) + \frac{\gamma w_u}{2}\sin(2w_u t) + w_0^2\right]$$

→ Take away: energy oscillates at twice the frequency of oscillation.

(How? Why?)



$\gamma \ll 2w_0$ (very lightly damped)

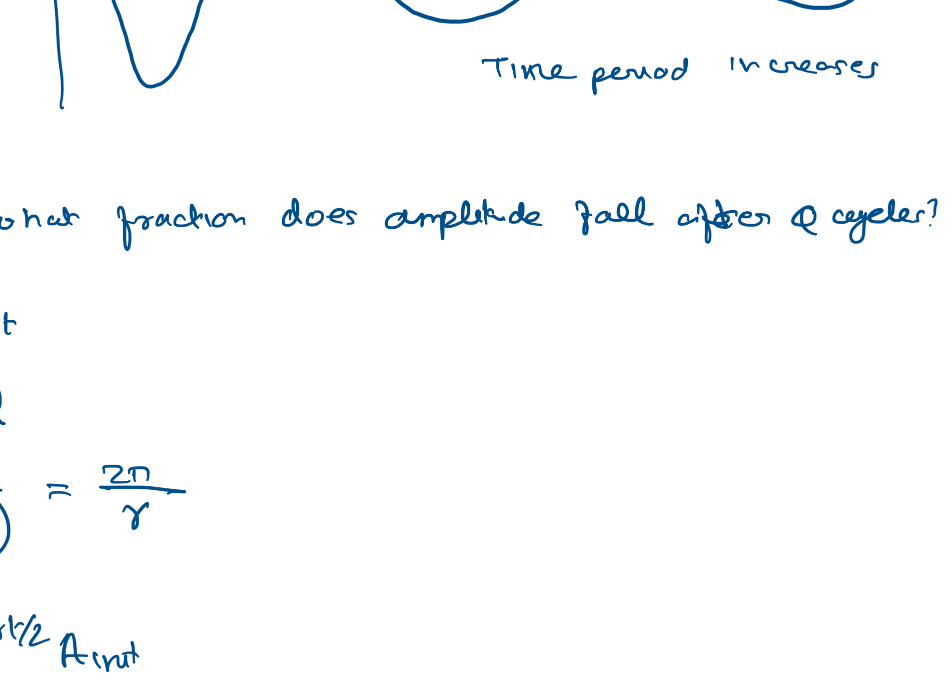
$$\langle E \rangle = \frac{1}{2}mw_0^2A^2e^{-\gamma t}$$

ignore oscillating term; energy without damping

Lets define some quantities you will encounter

$Q = w_0/\gamma$ } → dimensionless (its a quick check)

smaller Q
→ dies off quickly
→ oscillates with much longer time period



So, for $\gamma \ll 2w_0$, by what fraction does amplitude fall after Q cycles?

After Q cycles; $A_{final} = ?\,A_{init}$

$$w_u t = 2\pi Q$$

$$\Rightarrow t = \frac{2\pi}{(w_0/Q)} = \frac{2\pi}{\gamma}$$

$$\therefore A_{fin} = e^{-\gamma t/2}A_{init}$$

$$= e^{-\pi}A_{init}$$

independent of γ/w_0; 0.043

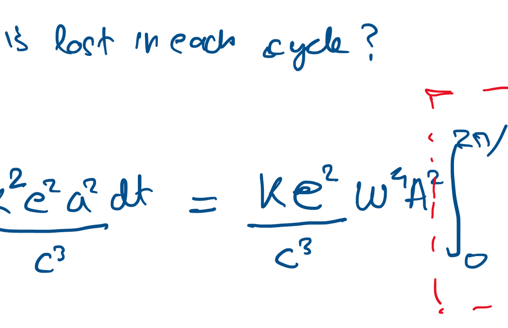
No of cycles after which amplitudes is 4.3% gives us Q.

Lets study a special case,

Classical EM: accelerating charges radiate energy → EM wave

We know from EM $\frac{dE}{dt} = \frac{Ke^2a^2}{c^3}$ } Power radiated.

Lets look at a really simple picture of atom: (consider very slightly damped)



For 1 cycle,

$x = A\sin(wt)$ for 1 coordinate

$a = \ddot{x} = -w^2x$

How much energy is lost in each cycle?

$$\Delta E = \int_0^{2\pi/w_0}\frac{K^2e^2a^2\,dt}{c^3} = \frac{Ke^2w^4A^2}{c^3}\underbrace{\int_0^{2\pi/w}\sin^2 wt\,dt}_{\pi/w}$$

energy lost in each cycle

What is the quality factor for the atom?

$$\frac{\Delta E}{E_{ini}} = -\gamma T \quad \text{from} \quad \langle E\rangle = \frac{1}{2}mw_0^2A^2e^{-\gamma t}$$

linear expanded

Plug in ΔE, $E_{ini} = \frac{1}{2}mw^2A^2$ & $T = 2\pi/w_0$

to get γ

Now $\tau = 1/\gamma$

or $Q = w_0/\gamma$

So if a e^- oscillates at a freq of green color ($\lambda \sim 500\,nm$) then we get $\tau = 10^{-8}$ secs. So how does the universal remain stable?

→ Learn in Ph2b.

Driven damped harmonic oscillator

$$\ddot{x} + \gamma\dot{x} + w_0^2x = \frac{F_0}{m}\cos wt$$

driving freq; generalized force; natural freq

Two behaviours → ① Transient → damping and natural frequencies all die out
② Steady state → only driving force remain. ($t\to\infty$)

We will solve for this next week.

But let's look at some diff eqn. fact

Inhomogenous Diff Equation

$$\ddot{y} + p(t)\dot{y} + q(t)y = g(t)$$ → more general 2nd order ODE

Maths tells me I can solve it in two parts

Homogenous

$$\ddot{y} + p(t)\dot{y} + q(t)y = 0$$

Say our solution is $y_c(t)$ ← complementary solution (we saw this already)

Inhomogenous

$y_p(t)$ solution ← particular (we will see this as we go ahead)

The claim is

$$y(t) = y_c(t) + y_p(t)$$ solve the inhomogenous eqn too by linearity.

So how do find $y_p(t)$ → Guess

e.g. $\ddot{y} - 4\dot{y} - 12y = 3e^{5t}$

guess $y_p(t) = Ae^{5t}$

$$25Ae^{5t} - 20Ae^{5t} - 12Ae^{5t} = 3e^{5t}$$

$$\Rightarrow A = -3/7$$

e.g. $\ddot{y} - 4\dot{y} - 12y = 2t^3 - t + 3$

guess $y_p(t) = At^3 + Bt^2 + Ct + D$ ← for nth order driving polynomial guess a similar polynomial

Plug & now match order by order

e.g. $g(t) = 5\cos 3t$

$y_p(t) = A\cos(3t) + B\sin(3t)$

This is called method of undetermined coefficients.