Lecture 3

Wednesday, October 5, 2022 9:13 AM

So far we studied SHO, but when is life simple.

So we look at damped harmonic oscillator.

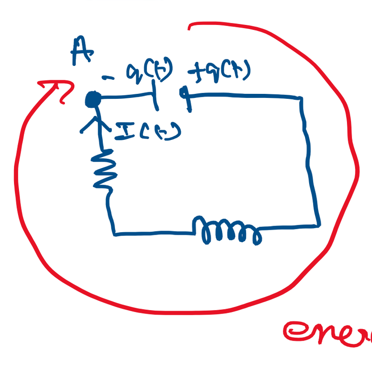
↳ we will basically add more layers of complexity as we go ahead

↳ solutions become more complex

We will build intuition as we go through this today.

Simple RLC circuit

We will take any direction as positive

→ 

energy conserved in conservative force

$$V_A + \frac{q}{C} - L\frac{dI}{dt} - IR = V_A$$

$$\Rightarrow L\frac{dI}{dt} - \frac{q}{C} + IR = 0$$

$$\Rightarrow L\frac{d\left(-\frac{dq}{dt}\right)}{dt} - q/C + IR = 0$$

$$\Rightarrow -L\frac{d^2q}{dt^2} - \frac{dq}{dt}R - q/C = 0$$

$$\Rightarrow \ddot{q} + \frac{R}{L}\dot{q} + \frac{q}{LC} = 0$$

Compare to $\ddot{q} = -\left(\frac{1}{LC}\right)q - \left(\frac{R}{L}\right)\dot{q}$

↳ extra damping term (cancels velocity)

Side Note

$$m\ddot{x} = -kx + F_{damp}(\dot{x})$$

first order $\dot{x} << 1$

$$\Rightarrow m\ddot{x} + b\dot{x} + kx = 0$$

$$\Rightarrow \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

$\gamma = R/L$, $\omega_0^2 = 1/LC$ | $\gamma = b/m$, $\omega_0^2 = k/m$ } These are viscous damping (not friction)

We will solve this most generic equation.

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = 0 \quad \& \quad x(t) = Re(z(t))$$

Trigonometry is hard, I am bad at it. Exp is just powers which add

How to solve a differential equation like this?

↳ Guess (Ansatz)

$$z(t) = Ce^{\alpha t}$$

$$\therefore C\alpha^2 e^{\alpha t} + \gamma C\alpha e^{\alpha t} + \omega_0^2 C e^{\alpha t} = 0$$

$$\Rightarrow \alpha^2 + \gamma\alpha + \omega_0^2 = 0 \quad \leftarrow \text{Characteristic equation}$$

$$\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$ } 3 scenarios

$$\therefore z(t) = \underbrace{C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}}$$

↳ Two constants or two solutions.

All thats fine, but how do I visualize it?

① $\gamma < 2\omega_0$ ← natural freq dominates

$$\alpha = -\frac{\gamma}{2} \pm i\omega_u$$

where $\omega_u = \frac{1}{2}\sqrt{4\omega_0^2 - \gamma^2} = \omega_0\sqrt{1 - \frac{\gamma^2}{4\omega_0^2}}$

$$\therefore z(t) = C_1 e^{-\gamma/2\, t} e^{+i\omega_u t} + C_2 e^{-\gamma/2\, t} e^{-i\omega_u t}$$

$$= e^{-\gamma/2\, t}\underbrace{\left(C_1 e^{i\omega_u t} + C_2 e^{-i\omega_u t}\right)}$$

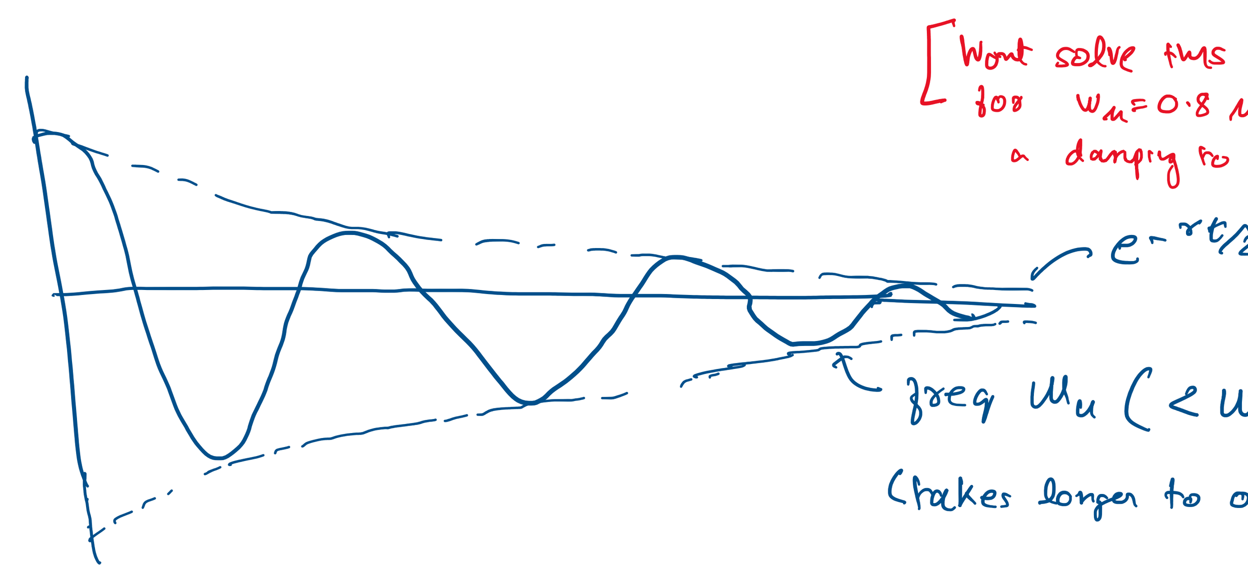
Does this remind you of something.

$$x(t) = 2e^{-\gamma/2\, t}\, C\cos(\omega_u t + \phi)$$

$$= \tilde{A}e^{-\gamma/2\, t}\cos(\omega_u t + \phi)$$

For imaginary part vanishing we need $C_1 = C_2^* = Ce^{i\phi}$

[Wont solve this but for $\omega_u = 0.8\,\omega_0$, we get a damping to 10^{-4} in 2 cycles]



(takes longer to oscillate)

② $\gamma > 2\omega_0$ → Overdamping (Sluggish)

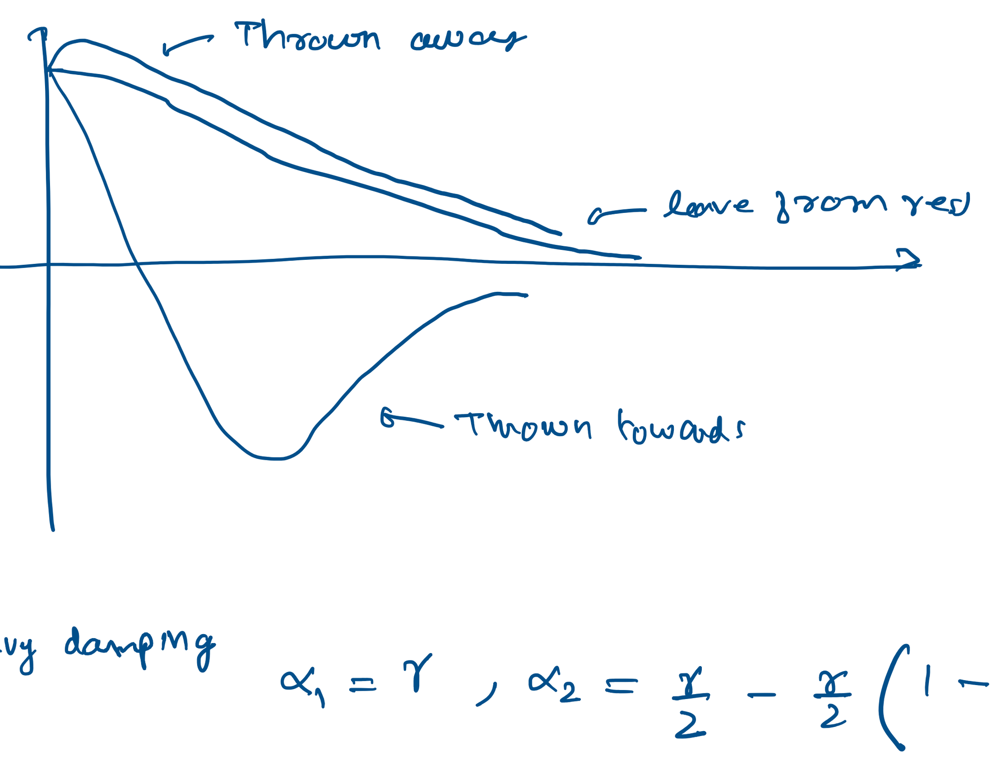
(no complex values) α is Re

$$\alpha_{1/2} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \quad \Rightarrow \quad \alpha_1 > \alpha_2 > 0$$

$$z_1(t) = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t}$$ } Who dominates motion in late term?

↳ dies out faster

$$\simeq C_2 e^{-\alpha_2 t}$$



Crosses origin at most once

$$C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t} = 0$$

$$t = -\frac{1}{\alpha_1 - \alpha_2}\ln\left(\frac{-C_1}{C_2}\right)$$

(no oscillations)

Heavy damping $\alpha_1 = \gamma$, $\alpha_2 = \frac{\gamma}{2} - \frac{\gamma}{2}\left(1 - \frac{4\omega_0^2}{2\gamma^2}\right)$

$$= +\omega_0^2/\gamma << \gamma$$

as $\gamma >> \omega_0$

At late time,

$$x(t) = C_2 e^{-(\omega_0^2/\gamma)t} = C_2 e^{-t/\tau}$$

$$\tau = \frac{\omega_0^2}{\gamma}$$

As you saw before $F_{damp} = -b\dot{x} = C_2 b\frac{\omega_0^2}{\gamma}e^{-\omega_0^2 t/\gamma}$

$$= -\frac{b\omega_0^2}{\gamma}x(t)$$

$$= -m\frac{k}{m}x(t)$$

$$= -kx$$

Damping force for very heavy damping matches the spring force.

③ Critical damping

$$\alpha_{1/2} = \gamma/2$$

$$\therefore z(t) = C_1 e^{-\gamma/2\, t} + C_2 t e^{-\gamma/2\, t}$$ } Take it from me

$$= (C_1 + C_2 t)e^{-\gamma/2\, t}$$

Similar curves as in overdamping but rate is different

α - n times

→ $e^{\alpha t}$, $te^{\alpha t}$, ..., $t^{n-1}e^{(n-1)\alpha t}$

Mass returns to the origin fastest } Car shock absorbers