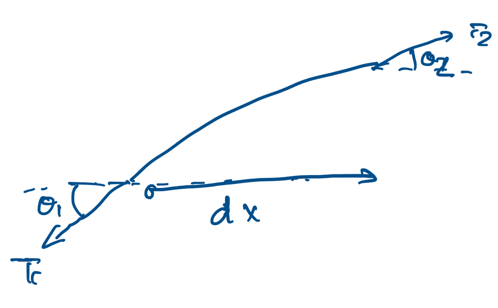
Lecture 10

Thursday, November 3, 2022 3:34 PM

For transverse wave like string waves, we can use



From $\frac{dy}{dx} = \tan\theta \simeq \theta$ as its small

Further as longitudinal forces must cancel, $T_1 = T_2$

For transverse forces,

$$F_{net} = T\sin\theta_2 - T\sin\theta_1$$
$$= T\left(y'(x+dx) - y'(x)\right)$$
$$= T dx \, \frac{\left(y'(x+dx) - y'(x)\right)}{dx}$$
$$= T dx \, \frac{d^2y}{dx^2}$$

$$\therefore \; m\frac{d^2y}{dt^2} = T dx \, \frac{d^2y}{dx^2}$$

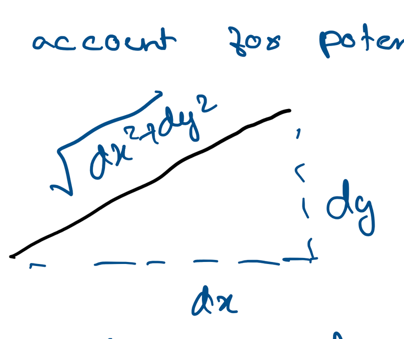
$$\Rightarrow \; \ddot{y} = \frac{T}{\mu} y''$$

$$y = Ae^{i(\pm kx \pm \omega t)}$$

We can also solve using energy,

$$K_{dx} = \frac{1}{2} dm \, V_y^2 = \frac{1}{2}(\mu dx)\dot{y}^2$$

But how do we account for potential energy:- (compare to springs)



$$\therefore \text{ Extended length} = dx\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} = dx + \frac{dx}{2}\left(\frac{\partial y}{\partial x}\right)^2$$

$$\therefore dl = \frac{dx}{2}\left(\frac{\partial y}{\partial x}\right)^2$$

$$\therefore \text{ Force } T \text{ does work} = T\,dl = \frac{1}{2} T dx \left(\frac{\partial y}{\partial x}\right)^2$$

$$\therefore \; \mathcal{E}(x,t) = \frac{K_{dx} + V_{dx}}{2} = \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial \psi}{\partial x}\right)^2$$
$$= \frac{\mu}{2}\left[\left(\frac{\partial \psi}{\partial t}\right)^2 + v^2\left(\frac{\partial \psi}{\partial x}\right)^2\right]$$

Also for travelling waves

$$\frac{\partial y}{\partial t} = \pm v\frac{\partial y}{\partial x} \quad \Big| \quad Z = \sqrt{T\mu} = \mu v$$

$$\therefore \; \mathcal{E}(x,t) = \mu\left(\frac{\partial y}{\partial t}\right)^2 \quad \& \quad \mathcal{E}(x,t) = \mu v^2\left(\frac{\partial y}{\partial x}\right)^2$$
$$= \frac{Z}{v}\left(\frac{\partial \psi}{\partial t}\right)^2 \qquad\qquad = Zv\left(\frac{\partial \psi}{\partial x}\right)^2$$

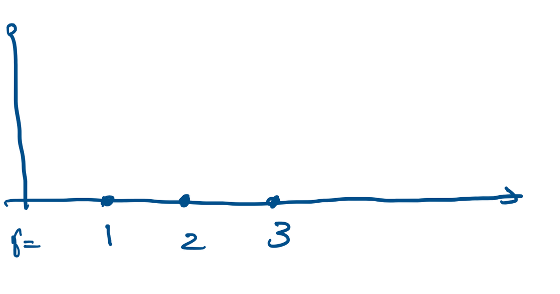
What is the power transmitted by an arbitrary wave?

$$P(x,t) = \frac{dW}{dt} = \frac{F_y\,dy}{dt} = F v_y = \left(-T\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)$$
$$= \mp \frac{T}{v}\left(\frac{\partial \psi}{\partial t}\right)^2$$
$$= \mp Z\left(\frac{\partial \psi}{\partial t}\right)^2$$
$$= \mp v\,\mathcal{E}(x,t)$$

Is there momentum? No, cm doesnt move

N coupled oscillators

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_N \end{pmatrix} = -M \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$



- $\rightarrow N \to \infty$
- $\rightarrow \Delta x \to 0$
- $\rightarrow N\Delta x = \text{fixed}$
- $\rightarrow \text{Mass} = N\Delta m = \text{fixed}$

In Continuum limit, $\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$ ← just from newton's laws

Travelling wave soln. $y(x,t) = f(x-vt) + g(x+vt)$

So what are my normal modes?

$$y_j = \tilde{A}_j e^{i\omega t} \quad \forall j$$

"j" labelled by x in continuum limit

$$\therefore \; y(x,t) = h(x)e^{i\omega t}$$

lets plug it in

$$(i\omega)^2 h(x)e^{i\omega t} = v^2 e^{i\omega t}\frac{\partial^2 h(x)}{\partial x^2}$$

$$\boxed{\Rightarrow \; \frac{\partial^2 h}{\partial x^2} = -\frac{\omega^2}{v^2}h(x)}$$

for linear dispersion

$$\underbrace{\frac{\partial^2 h}{\partial x^2} = -k^2 h(x)}_{\text{simple harmonic oscillator}} \qquad k = \frac{\omega}{v} = \frac{2\pi}{\lambda}$$

$$h(x) = A\cos kx + B\sin kx$$

$\therefore$ N Modes :

$$y(x,t) = h(x)\left(C\cos(\omega t) + D\sin(\omega t)\right)$$
$$= \left(A\cos kx + B\sin kx\right)\left(C\cos(\omega t) + D\sin(\omega t)\right)$$

∞ such modes, as we can find a solution for every $\omega \in \mathbb{R}^+$

(uncountably infinite)

<u>NM + Boundary Condns</u>

$y(0,t) = 0$ ← fixed end

$y(L,t) = 0$ ← fixed end

What happens to my normal modes?

lets just take $t=0$

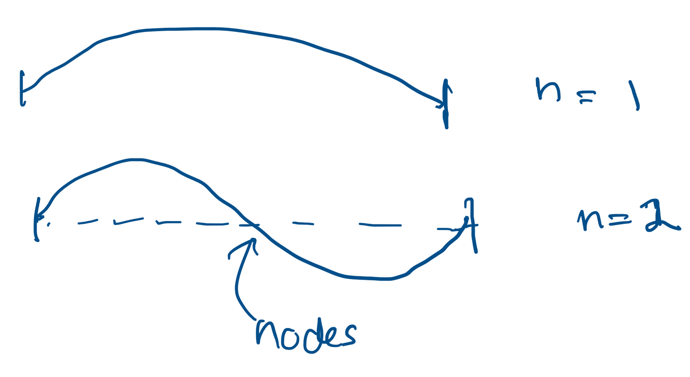
$$y(x,0) = 0 = \left(A\cos kx + B\sin kx\right)C$$
$$\Downarrow$$
$0 = A$ ← at $x = 0$

$0 = B\sin(kL)$ ← at $x = L$

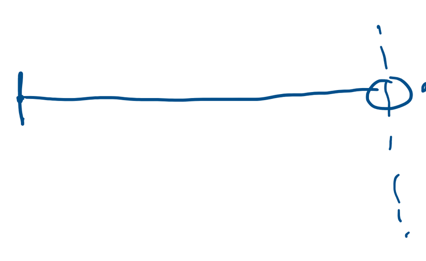
$$\therefore \; kL = n\pi \Rightarrow k_n = \frac{n\pi}{L}$$

Quantization

$$\therefore \; \omega_n = k_n v = \frac{n\pi v}{L}$$ ← countably infinite normal modes.



What if only one end is fix



$\Delta m \to 0$, so atleast $F \to 0$

$\frac{dy}{dx} \to 0$

$$B\sin(kL)k = 0$$
$$k_n L = \left(n + \frac{1}{2}\right)\pi$$

Lets go back once more,

$$y_n(x,t) = \left(A_n\cos k_n x + B_n\sin k_n x\right)\left(C_n\cos\omega_n t + D_n\sin\omega_n t\right)$$

$$y(x,t) = \sum_n y_n(x,t)$$

(1) String at $t=0$, has zero speed $\frac{\partial y}{\partial t}(x,0) = 0 \Rightarrow D_n = 0$

$$y(x,t) = \left(A_n\cos(k_n x) + B_n\sin(k_n x)\right)\cos\omega_n t$$

$$y(x,0) = \text{known} = p(x)$$

$$\underbrace{p(x) = \sum_n B_n\sin\left(\frac{n\pi x}{L}\right)}_{\text{We will see this later}}$$

This is fourier expansion