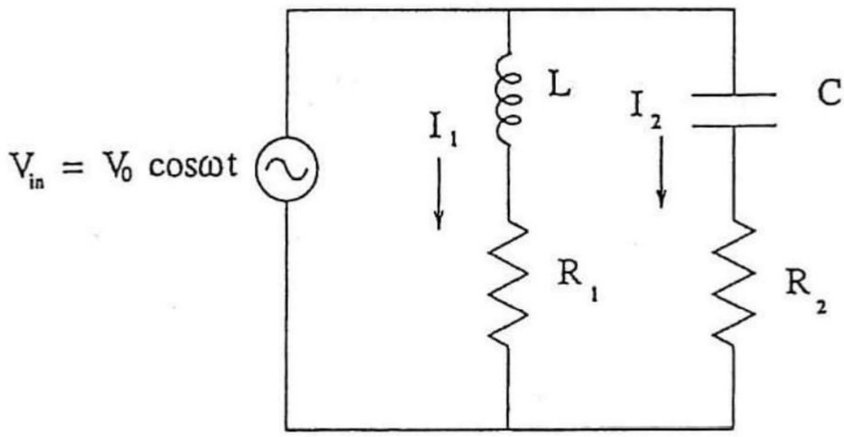
AC Circuits
Problem 1

The alternating current generator in the figure below supplies a voltage $V_{in} = V_0 \cos \omega t$, resulting in currents I_1 and I_2 flowing through the parallel branches of the circuit shown below.



(a) Find the complex impedances Z_1 and Z_2 of the parallel branches of the circuit.

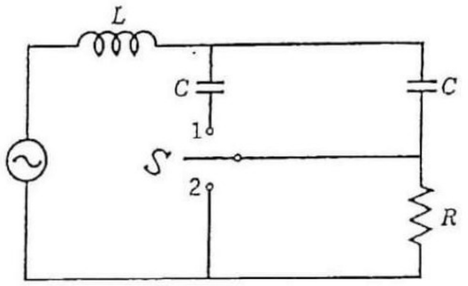
(b) Find the amplitude (modulus), and the phase relative to V_{in}, of the currents I_1 and I_2.
(Use the phase convention $\hat{I}(t) = I_0 e^{i(\omega t+\varphi)}$).

(c) Find expressions for the average power P_1 delivered to resistor R_1 (averaged over a cycle), and the average power P_2 delivered to resistor R_2 (also averaged over a cycle) as a function of ω, and sketch them. Find the values ω_1 and ω_2 at which P_1 and P_2 are at half their maxima, respectively. Mark these points on your sketch.

(d) For the case $R_1 = R_2$, find the frequency ω_c at which $P_1 = P_2$.

Problem 2

The circuit shown in the illustration is driven by a generator that supplies an alternating EMF of the form $V(t) = V_0 \cos \omega t$. The current through the resistor can be expressed as $I(t) = I_0 \cos(\omega t + \phi)$, where I_0 and ϕ depend on the position of the switch $\mathcal{S}$.



(a) For a fixed V_0, find the angular frequency ω_* for which the current I has maximal amplitude for the switch in position 1. What is the value of the amplitude I_0 when $\omega = \omega_*$?

(b) For an arbitrary angular frequency ω, what is the complex impedance Z_0 when the switch is open?

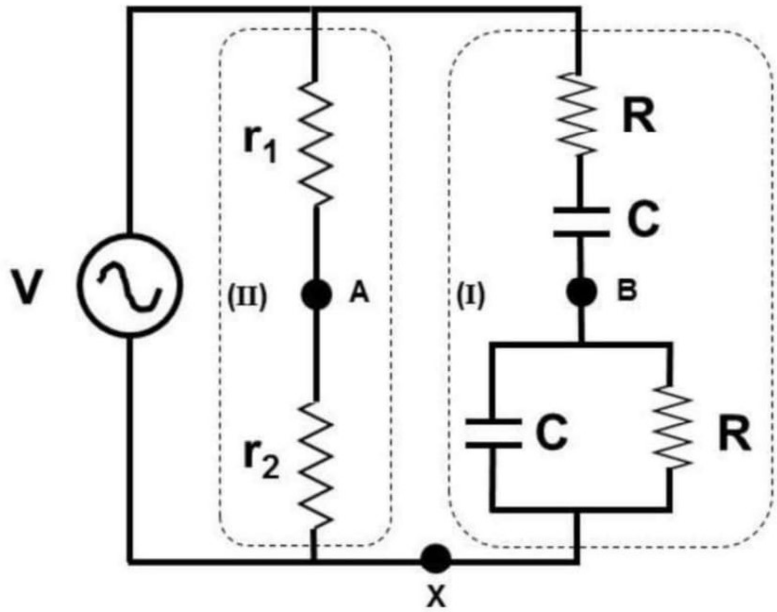
(c) What is the impedance Z_1 when the switch is in position 1?

(d) What is the impedance Z_2 when the switch is in position 2?

Now suppose that the generator provides an rms voltage of 220 V, at a frequency $\nu = 50$ Hz. When the switch is open, the current I leads the driving voltage V by a phase $\phi = 45°$. When the switch is in position 1, the current lags behind the voltage by a phase $\phi = -26.6°$. When the switch is in position 2, the rms current is 2 Amps.

(e) Find the values of R, L, and C.

Problem 3



The circuit above is called a Wien bridge, frequently used in RC oscillators, and consists of two branches (I) and (II) that are enclosed by dashed lines. The voltage source produces an AC voltage with amplitude V and angular frequency ω. Express your answers below in terms of these and other constants given in the diagram.

a) Calculate the impedance Z_1 of the right branch (I) of the circuit.

b) Calculate the amplitudes of the currents $|I_1|$ and $|I_2|$ in branches (I) and (II).

c) Find the magnitude of the potential difference between points B and X.

A current meter is now connected between points A and B.

d) The circuit is said to be "balanced" when no current flows through the meter. Find the ratio r_1/r_2 and the frequency ω_0 for which the circuit is balanced.

e) What is the average power dissipation through branch (I) and through branch (II) under the balanced condition?

Displacement Current

Problem 4

Consider a parallel plate capacitor of capacitance C where the two plates are disks of radius b. Any physical capacitor slowly leaks current over time, even when disconnected from a circuit, as charges move between the two plates. We can model this effect by assuming the material between the two plates has a large resistance R.

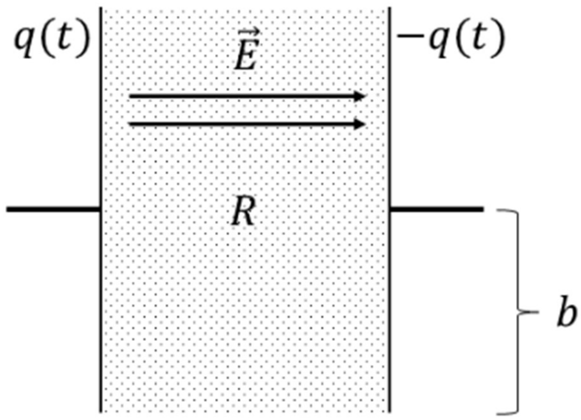
(a) Let $q(t)$ be the charge on the positive plate of the capacitor as a function of time. If the capacitor initially has charge q_0, find the charge as a function of time.

(b) What is the current as a function of time? Assume the current travelling between the two plates is uniform. What is the current density?

(c) What is the electric field in the material as a function of time? Assume the field is uniform and neglect edge effects resulting from the fact that the plates are finite.

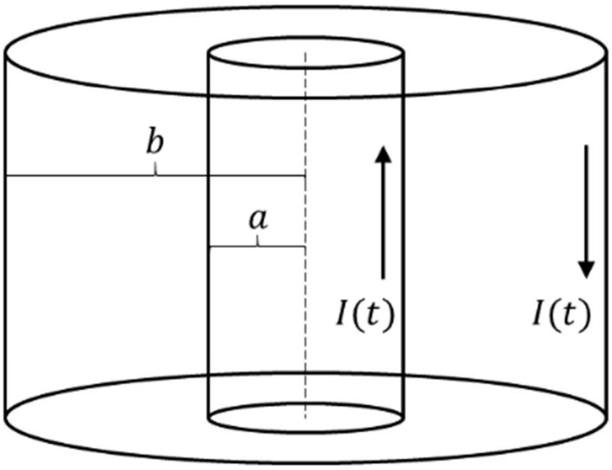
(d) What is the displacement current density?

(e) Does this discharging process induce a magnetic field?



Problem 5

Consider a coaxial cable consisting of two concentric cylinders. The inner cylinder has radius a while the outer cylinder has radius b. A current $I(t) = I_0 \cos(\omega t)$ travels along the inner cylinder and returns along the outer cylinder in the opposite direction. Assume this current is uniformly distributed across the two cylinders.



(a) Using Ampere's Law, find the magnetic field induced by the current.

(b) As this magnetic field is time dependent, find the electric field induced.

(c) What is the displacement current density, J_d?

(d) Let's now assume that the inner cable is very thin so $a \to 0$. What is the total displacement current, I_d? (Hint: $\int r \ln(r) dr = \frac{1}{4} r^2 [2 \ln(r) - 1]$)

(e) Now consider a typical coaxial cable. We will approximate $b \approx 3\,\text{mm}$ and $c = 3 \cdot 10^8\,\frac{\text{m}}{\text{s}}$. For what frequency is $\frac{I_d}{I} \approx \frac{1}{100}$? The calculation does not need to be exact, just accurate to a factor ten. What frequency band does this occur in (e.g. megahertz, gigahertz, terahertz, or higher)?

Problem 6

Up until now, you have simply been told that the power dissipated in a resistor is given by $P = IV$. We will now prove this result by making use of the Poynting vector (or at least "prove" this to the satisfaction of a physicist). Consider an infinitely long wire of radius a carrying a current I. For a segment of the wire of length L, the voltage difference between the two ends is V_0.

(a) Assume that the electric field inside of the wire is uniform. Find the electric field inside the segment of wire in terms of the voltage difference.

(b) What is the magnetic field induced outside of the wire?

(c) Assume that the expressions for the magnetic field and electric field are both valid along the surface of the wire. What is the Poynting vector at the surface of the wire? What does the direction of the Poynting vector tell us about the flow of energy?

(d) How much power is dissipated by the segment of wire?

EM Waves

Problem 7

Consider the following set of electric and magnetic fields which correspond to a travelling plane wave:

$$
\begin{aligned}
E_x &= E_0 \sin(kz - \omega t) \qquad & B_x &= 0 \\
E_y &= 0 & B_y &= \left(\tfrac{E_0}{c}\right) \sin(kz - \omega t) \\
E_z &= 0 & B_z &= 0
\end{aligned}
$$

(a) Show that these electric and magnetic fields satisfy Maxwell's equations if k and ω are related in a certain way.

(b) What direction is this wave travelling in? (Hint: Consider examining a particular point on the wave such as a maximum or minimum and examining how this point moves over time.)

(c) What is the Poynting vector of this wave? How does this relate to the direction of travel?

(d) Consider a loop of radius R located at the origin with normal vector along the $\hat{z}$ direction. What is the instantaneous power passing through this loop? What is the average power?

Problem 8

Sunlight with B-field $\vec{B} = 10^{-6}(B_x, 2, 0)\cos(\omega t - k_x x + k_y y)$ webers/m^2, where $k_x = 8\pi/a$, $k_y = 6\pi/a$ and a = 2500 nanometers, shines on a mirror-like polar ice lying in the x-z plane. The area of the ice is 100,000 square kilometers.

1. Find B_x, the direction, the wavelength and the frequency of the incident sunshine;
2. Find the E-field of the incident sunshine;
3. Find the Poynting vector of the incident sunshine;
4. Find the reflected E-field;
5. Find the total E-field;
6. Due to global warming, this piece of ice has just melted, exposing the totally absorbing black rocks underneath. How much additional energy per second is the Earth absorbing due to this melted ice? How does it compare with the average electrical power consumption of the world (about four times the US power consumption, which is the average power consumption per person times the US population)?

Problem 9

(a) (4 Points) Let us consider a traveling EM wave with electric field given by

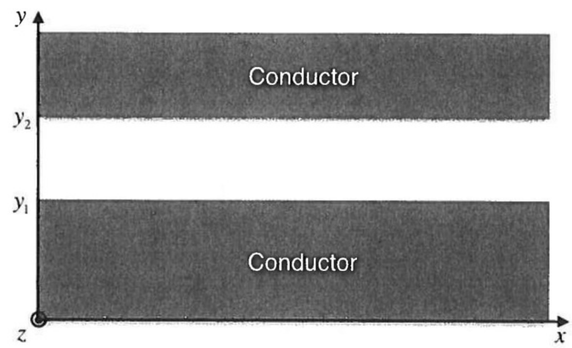
$$\mathbf{E}_1 = E_0\hat{\mathbf{z}}\sin(\omega t - k_x x - k_y y), \quad \mathbf{B}_1 = \mathbf{b}_1 \sin(\omega t - k_x x - k_y y), \tag{1}$$

where E_0, ω, k_x and k_y are real-valued constants, and $\mathbf{b}_1$ is a constant vector. Given E_0, k_x and k_y, find the $\mathbf{b}_1$ and ω that make $\mathbf{E}$ and $\mathbf{B}$ together satisfy the Maxwell Equations. Show that this is a plane wave. What are the frequency and the wavelength? What is the direction of propagation? Compute the Poynting vector.

(b) (1 Point) For the same ω, k_x, k_y and E_0 above, consider another set of fields:

$$\mathbf{E}_2 = E_0\hat{\mathbf{z}}\sin(\omega t - k_x x + k_y y), \quad \mathbf{B}_2 = \mathbf{b}_2 \sin(\omega t - k_x x + k_y y). \tag{2}$$

Find the constant vector $\mathbf{b}_2$ required for $\mathbf{E}_2$ and $\mathbf{B}_2$ to satisfy Maxwell Equations.



(c) (2 Points) Consider the total fields given by the superposition of the above two solutions,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \quad \mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2. \tag{3}$$

Given a specific set of k_x and k_y, suppose we place perfect conductors for half spaces $y < y_1$ and $y > y_2$ (as shown in the figure), and require that Eq. (3) can describe the wave in between, i.e., $y_1 < y < y_2$. In order for $\mathbf{E}_\parallel = 0$ on the boundary, what are the allowed values for y_1 and y_2?

Hint: you may want to show that E_z is of the following form

$$E_z(t,x,y,z) = f(t,x)g(y) \tag{4}$$

(d) (1 Point) Check that $\mathbf{B}_\perp = 0$ is also satisfied by the allowed values in part (c).

(e) (1 Points) Given $y_2 - y_1 = a > 0$, what are the allowed values for k_y? Using this result, together with part (a) and the fact that k_x is real valued, show that there is a minimum value for ω such that the solution in Part (c) can exist.

(f) (1 Points) Show that the total $\mathbf{E}$ and $\mathbf{B}$ fields move along the x direction, and compute the phase velocity (i.e., velocity at which the field patterns move). *[Hint: use Eq. (4) and note that phase velocity can sometimes be larger than the speed of light in vacuum.]*

Solution 1

(a) $Z_1 = i\omega L + R_1$, $Z_2 = \frac{1}{i\omega C} + R_2$

(b) $I_1 = \frac{V_0 e^{i\omega t}}{Z_1} = \frac{V_0 e^{i\omega t}}{i\omega L + R_1}$, so $|I_1| = \frac{V_0}{\sqrt{(\omega L)^2 + {R_1}^2}}$, $\varphi_1 = -arctan\frac{\omega L}{R_1}$

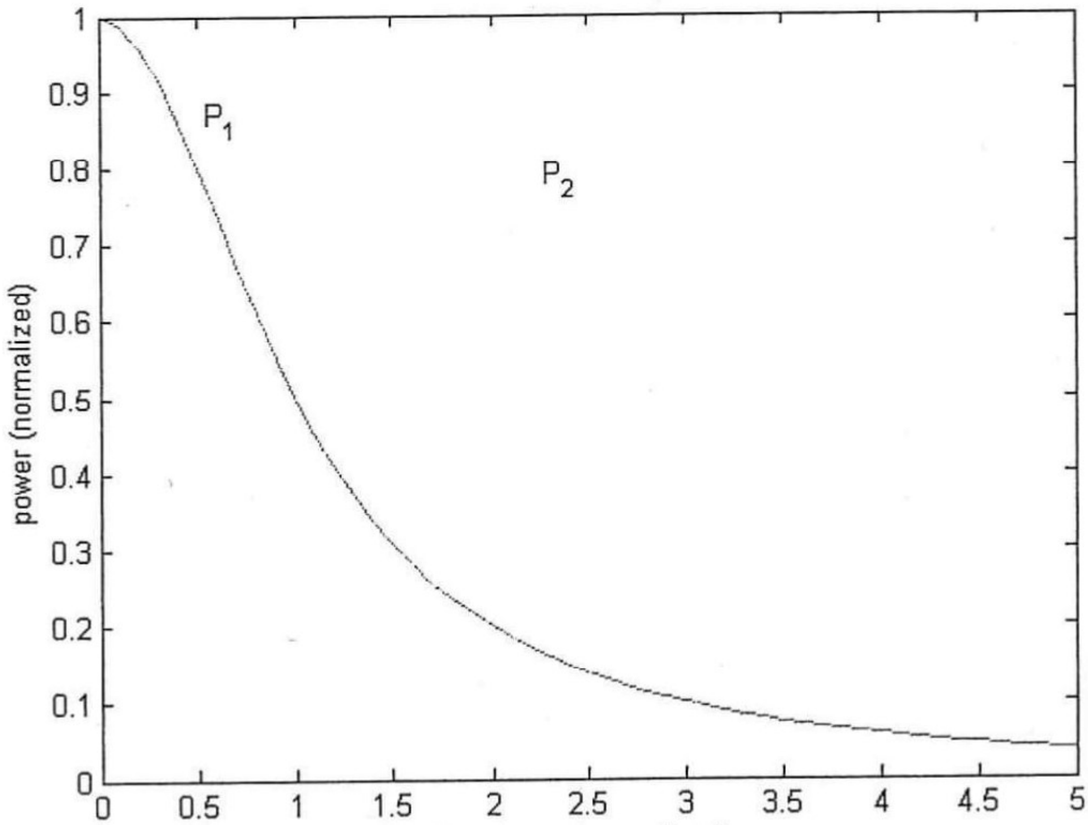
$I_2 = \frac{V_0 e^{i\omega t}}{Z_2} = \frac{V_0 e^{i\omega t}}{\frac{1}{i\omega C} + R_2}$, $|I_2| = \frac{V_0}{\sqrt{(\frac{1}{i\omega C})^2 + {R_2}^2}}$, $\varphi_1 = arctan\frac{1}{\omega C R_2}$

(c) $< P_1 > = \left(\frac{1}{\sqrt{2}}V_0\right)\left(\frac{1}{\sqrt{2}}|I_1|\right) cos\varphi_1 = \frac{1}{2}{V_0}^2 \frac{R_1}{(\omega L)^2 + {R_1}^2}$, maximum at $\omega = 0$.

When $< P_1 > = \frac{1}{2} < P_1 >_{max}$, $(\omega_1 L)^2 + {R_1}^2 = 2{R_1}^2$, $\omega_1 = \frac{R_1}{L}$

and $< P_2 > = \left(\frac{1}{\sqrt{2}}V_0\right)\left(\frac{1}{\sqrt{2}}|I_2|\right) cos\varphi_2 = \frac{1}{2}{V_0}^2 \frac{R_2}{(\frac{1}{\omega C})^2 + {R_2}^2}$, maximum at $\omega = \infty$.

When $< P_2 > = \frac{1}{2} < P_2 >_{max}$, $(\frac{1}{\omega_2 C})^2 + {R_2}^2 = 2{R_2}^2$, $\omega_2 = \frac{1}{CR_2}$



(d) $\frac{1}{2}{V_0}^2 \frac{R_1}{(\omega L)^2 + {R_1}^2} = \frac{1}{2}{V_0}^2 \frac{R_2}{(\frac{1}{\omega C})^2 + {R_2}^2}$, $R_1 = R_2$

so $\omega L = \frac{1}{\omega C}$, $\omega_c = \frac{1}{LC}$

Solution 2

(a) For a fixed V_0, find the angular frequency ω_* for which the current I has maximal amplitude for the switch in position 1. What is the value of the amplitude I_0 when $\omega = \omega_*$?

Solution: If the switch $\mathcal{S}$ is in position 1, then the two capacitors C are connected in parallel. The total impedance of the circuits is:

$$Z_1 = i\omega L - \frac{1}{2}\frac{i}{\omega C} + R = R + i\left(\omega L - \frac{1}{2\omega C}\right) .$$

The term in parenthesis vanishes for an angular frequency

$$\omega_*^2 = \frac{1}{2LC} \quad \Rightarrow \quad \underline{\omega_* = \frac{1}{\sqrt{2LC}}} .$$

This is clearly the frequency that minimizes $|Z_1|$, and therefore also the frequency the maximizes the current amplitude $I_0 = V_0/|Z_1|$.

The value of I_0 when $\omega = \omega_*$ is simply $\underline{V_0/R}$.

(b) For an arbitrary angular frequency ω, what is the complex impedance Z_0 when the switch is open?

Solution: When the switch $\mathcal{S}$ is open, the capacitor C that connects to switch position 1 is irrelevant, and the impedance of the circuit is:

$$Z_0 = i\omega L - \frac{i}{\omega C} + R = R + i\left(\omega L - \frac{1}{\omega C}\right) .$$

(c) What is the impedance Z_1 when the switch is in position 1?

Solution: We have already computed this impedance in the solution to part **(a)**:

$$\underline{Z_1 = R + i\left(\omega L - \frac{1}{2\omega C}\right)} .$$

(d) What is the impedance Z_2 when the switch is in position 2?

Solution: When the switch $\mathcal{S}$ is in position 2, the resistor R is shorted out and the capacitor C connected to switch position 1 is irrelevant. Therefore the impedance of the circuit is:

$$Z_2 = i\omega L - \frac{i}{\omega C} = \underline{i\left(\omega L - \frac{1}{\omega C}\right)} .$$

Now suppose that the generator provides an rms voltage of 220 V, at a frequency $\nu = 50$ Hz. When the switch is open, the current I leads the driving voltage V by a phase $\phi = 45°$. When the switch is in position 1, the current lags behind the voltage by a phase $\phi = -26.6°$. When the switch is in position 2, the rms current is 2 Amps.

(e) Find the values of R, L, and C.

Solution: First, we convert from period frequency $\nu = 50$ Hz to the equivalent angular frequency $\omega = 2\pi\nu = 100\pi$ rad/s.

For switch $\mathcal{S}$ open:

Using the result from part **(b)** and the formula

$$\tan\phi = -\frac{\Im(Z)}{\Re(Z)} ,$$

we have that

$$\frac{1}{\omega C} - \omega L = R . \qquad [*]$$

For switch $\mathcal{S}$ in position 1:

Using

$$\tan\phi = -\frac{1}{2} = \left(\frac{1}{2\omega C} - \omega L\right)/R ,$$

we have that

$$\frac{1}{2\omega C} - \omega L = -\frac{R}{2} . \qquad [**]$$

We can solve equations [*] and [**] for L and C in terms of R and ω:

$$\begin{aligned} C &= \frac{1}{3\omega R} \\ L &= \frac{2R}{\omega} . \end{aligned}$$

For the switch $\mathcal{S}$ in position 2:

We have that

$$I_{rms} = \frac{V_{rms}}{|Z_2|} = \frac{V_{rms}}{\frac{1}{\omega C} - \omega L} .$$

Using equation [*] we have that

$$\begin{aligned} I_{rms} &= \frac{V_{rms}}{R} \\ 2\ \text{A} &= \frac{220\ \text{V}}{R} \qquad \Rightarrow \qquad \underline{R = 110\ \Omega} . \end{aligned}$$

Therefore

$$C = \frac{1}{3\omega R} = \underline{\frac{1}{3(100\pi)(110)} = 9.65\ \mu\text{F}} ,$$

and

$$L = \frac{2R}{\omega} = \underline{\frac{220}{100\pi} = 0.7\ \text{H}} .$$

Notice that these values are in SI units. You can convert them to CGS units by using the equivalences:

$$\begin{aligned} 1\ \Omega &= 1.113 \times 10^{-12}\ \text{cm}^{-1}\text{s} \\ 1\ \text{F} &= 9 \times 10^{11}\ \text{cm} \\ 1\ \text{H} &= 1.113 \times 10^{-12}\ \text{cm}^{-1}\text{s}^{-2} . \end{aligned}$$

Solution 3

(a) The impedance of the right branch (I) of the circuit is given by

$$Z_1 = R + \frac{1}{i\omega C} + \frac{1}{1/R + i\omega C} = R\left(1 + \frac{1}{1+\omega^2 R^2 C^2}\right) - i\left(\frac{1}{\omega C} + \frac{\omega R^2 C}{1+\omega^2 R^2 C^2}\right)$$

(b) The amplitudes of the currents in branches (I) and (II) are given by

$$|I_1| = \frac{V}{|Z_1|} = \omega V C \sqrt{\frac{1+\omega^2 R^2 C^2}{1+7\omega^2 R^2 C^2 + \omega^4 R^4 C^4}} \;, \quad |I_2| = \frac{V}{r_1 + r_2} \,.$$

(c) The magnitude of the potential difference between points B and X is calculated as

$$\Delta V_{BX} = |I_1||Z_{BX}| = \omega V C \sqrt{\frac{1+\omega^2 R^2 C^2}{1+7\omega^2 R^2 C^2 + \omega^4 R^4 C^4}} \left|\frac{1}{1/R + i\omega C}\right|$$

$$= \frac{\omega V C R}{\sqrt{1+7\omega^2 R C^2 + \omega^4 R^4 C^4}} \,.$$

(d) When the circuit is balanced, we have

$$\frac{r_1}{r_2} = \frac{R + \dfrac{1}{i\omega_0 C}}{\dfrac{1}{1/R + i\omega_0 C}} = 2 + i\left(\omega_0 RC - \frac{1}{\omega_0 RC}\right) .$$

Since r_1/r_2 is real, then we require

$$\omega_0 RC - \frac{1}{\omega_0 RC} = 0 \Rightarrow \omega_0 = \frac{1}{RC} \,,$$

and $r_1/r_2 = 2$.

(e) From (a) and (d), the impedance Z_1 under the balanced condition ($\omega RC = 1$) is given by

$$Z_1 = \frac{3R}{2}(1 - i) \,.$$

It follows that

$$\phi_1 = \arg(Z_1) = -\frac{\pi}{4} \,.$$

The average power dissipation through branch (I) is then

$$P_1 = \frac{1}{2} V |I_1|_0 \cos(-\pi/4) = \frac{1}{2} V \times \frac{V}{R}\sqrt{\frac{1+1}{1+7+1}} \times \frac{\sqrt{2}}{2} = \frac{V^2}{6R} \,.$$

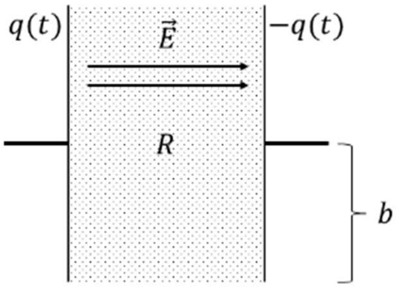
For branch (II), it rather easy,

$$P_2 = \frac{1}{2} V |I_2|_0 \cos(0) = \frac{V^2}{6 r_2} \,,$$

here we have used $r_1 = 2r_2$.

Solution 4

Consider a parallel plate capacitor of capacitance C where the two plates are disks of radius b. The material between the two plates has a large resistance R. We will declare the direction from the positive plate to the negative plate to be $\hat{z}$.



(a) Using Kirchoff's Laws, one has,

$$0 = \frac{q(t)}{C} + RI(t)$$
$$0 = \frac{1}{C}q(t) + R\frac{dq(t)}{dt}$$
$$\frac{dq(t)}{dt} = -\frac{1}{RC}q(t)$$

Solving the differential equation,

$$q(t) = q_0 e^{-\frac{t}{RC}}$$

(b) We define positive current as flowing from the positive plate to the negative plate. Then,

$$I(t) = -\frac{dq}{dt} = \frac{q_0}{RC}e^{-\frac{t}{RC}}$$

Assume the current travelling between the two plates is uniform, the current density is just the total current divided by the area. So,

$$\vec{J}(t) = \frac{I(t)}{\pi b^2}\hat{z} = \frac{q_0}{\pi b^2 RC}e^{-\frac{t}{RC}}\hat{z}$$

(c) The charge density of each plate is given by $\sigma = \pm\frac{q(t)}{\pi b^2}$. Treating the plates as if they were infinite,

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z}$$
$$= \frac{q_0}{\pi b^2 \epsilon_0}e^{-\frac{t}{RC}}\hat{z}$$

(d) The displacement current density is defined as $\vec{J_d}(t) = \epsilon_0 \frac{\partial}{\partial t}\vec{E}(t)$. Then, we find,

$$\vec{J_d}(t) = -\frac{q_0}{\pi b^2 RC}e^{-\frac{t}{RC}}\hat{z}$$

(e) For magnetic fields, Maxwell's equations state $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{B} = \mu_0\left(\vec{J} + \vec{J_d}\right)$. Examining the curl for this system,

$$\nabla \times \vec{B} = \mu_0\left(\vec{J} + \vec{J_d}\right)$$
$$\nabla \times \vec{B} = \mu_0\left(\frac{q_0}{\pi b^2 RC}e^{-\frac{t}{RC}} - \frac{q_0}{\pi b^2 RC}e^{-\frac{t}{RC}}\right)\hat{z}$$
$$\nabla \times \vec{B} = 0$$

Thus, no magnetic field is produced while the capacitor is leaking charge.

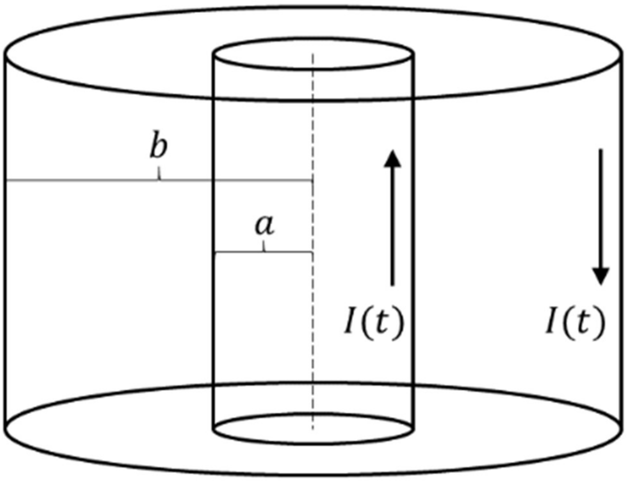
Solution 5

Consider a coaxial cable consisting of two concentric cylinders. The inner cylinder has radius a while the outer cylinder has radius b. A current $I(t)$ travels along the inner cylinder and returns along the outer cylinder in the opposite direction. Assume this current is uniformly distributed across the two cylinders. We define the $\hat{z}$ direction to be along the direction of the current of the inner cylinder.

(a) Ampere's Law states $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$. If we consider Amperian loops centered about the axis of the cylinder with radius $r < a$ or $r > b$, no net current is enclosed. Thus there is no magnetic field except between the two layers of the coaxial cable. Considering a loop of radius $a < r < b$,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$
$$B_\theta 2\pi r = \mu_0 I(t)$$
$$\vec{B} = \frac{\mu_0 I_0 \cos(\omega t)}{2\pi r}\hat{\theta}$$

(b) From the differential form of Faraday's Law, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. The electric field must be perpendicular to the magnetic field and by symmetry we expect the electric field to be in the $\hat{z}$ direction. We will consider a loop in the r-z plane. The loop will have length l along the z-axis and will go from some



radius $a < r_1 < b$ out to some other radius $b < r_2$. Then, using Faraday's Law,

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{A}$$
$$E_z l = -\frac{\partial}{\partial t} \int_{r_1}^{b} \frac{\mu_0 I_0 \cos(\omega t)}{2\pi r} l\, dr$$
$$E_z = -\frac{\partial}{\partial t} \frac{\mu_0 I_0 \cos(\omega t)}{2\pi} \ln\left(\frac{b}{r_1}\right)$$
$$\vec{E} = \frac{\mu_0 I_0 \omega \sin(\omega t)}{2\pi} \ln\left(\frac{b}{r}\right)\hat{z}$$

(c) The displacement current density is defined as $\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. Thus,

$$\vec{J}_d = \frac{\epsilon_0 \mu_0 I_0 \omega^2 \cos(\omega t)}{2\pi} \ln\left(\frac{b}{r}\right)\hat{z}$$

(d) To find the total displacement current, I_d, we integrate the displacement current density over its cross-sectional area. So,

$$\begin{aligned}
I_d &= \iint \vec{J}_d \cdot d\vec{A} \\
&= \int_0^b dr 2\pi r \frac{\epsilon_0 \mu_0 I_0 \omega^2 \cos(\omega t)}{2\pi} \ln\left(\frac{b}{r}\right) \\
&= \epsilon_0 \mu_0 I_0 \omega^2 \cos(\omega t) \int_0^b dr \left[r \ln(b) - r \ln(r)\right] \\
&= \frac{\omega^2}{c^2} I_0 \cos(\omega t) \left[\frac{1}{2} b^2 \ln(b) - \frac{1}{4} b^2 \left(2\ln(b) - 1\right)\right] \\
I_d(t) &= \frac{1}{4}\left(\frac{\omega b}{c}\right)^2 I_0 \cos(\omega t)
\end{aligned}$$

(e) We note that $I(t) = I_0 \cos(\omega t)$. So,

$$\frac{I_d(t)}{I(t)} = \frac{1}{4}\left(\frac{\omega b}{c}\right)^2$$

Plugging in $\frac{I_d(t)}{I(t)} = \frac{1}{100}$, $b = 3\,\text{mm}$, and $c = 3 \cdot 10^8\,\frac{\text{m}}{\text{s}}$ we find,

$$\begin{aligned}
\frac{1}{100} &= \frac{1}{4}\left(\frac{\omega \cdot 3 \cdot 10^{-3}\,\text{m}}{3 \cdot 10^8\,\frac{\text{m}}{\text{s}}}\right)^2 \\
\frac{1}{5} &= \left(\omega \cdot 10^{-11}\,\text{s}\right) \\
\omega &= 2 \cdot 10^{10}\,\frac{1}{\text{s}}
\end{aligned}$$

This is the angular frequency, if one would rather talk of the cycles per second, $f = \frac{2}{2\pi} \cdot 10^{10}\,\text{Hz} \approx 3.2\text{GHz}$. In any case, this occurs in the gigahertz range of frequencies. At these frequencies, one might need to start worrying about the effects of the displacement current. However, if the electrical current in the coaxial cable is switching at lower frequencies, the displacement current can effectively be ignored.

Solution 6

Consider an infinitely long wire of radius a carrying a current I. For a segment of the wire of length L, the voltage difference between the two ends is V_0. We choose to work in a coordinate system where $\hat{z}$ is the direction in which the current flows.

(a) Recall that the electric potential difference between two points and the electric field are related by,

$$V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l}$$

Thus, assuming that the electric field is uniform, we have,

$$\begin{aligned} -V_0 &= -\int_0^L E_z dz \\ \vec{E} &= \frac{V_0}{L}\hat{z} \end{aligned}$$

(b) We will consider a circular Amperian loop about the axis of the wire. Then, using Ampere's Law,

$$\begin{aligned} \oint \vec{B} \cdot d\vec{l} &= \mu_0 I_{\text{enc}} \\ B_\theta 2\pi r &= \mu_0 I \\ \vec{B} &= \frac{\mu_0 I}{2\pi r}\hat{\theta} \end{aligned}$$

(c) The Poynting vector is defined as $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}$. Thus,

$$\begin{aligned} \vec{S} &= \frac{\vec{E}\times\vec{B}}{\mu_0} \\ &= \frac{1}{\mu_0}\left(\frac{V_0}{L}\hat{z}\right) \times \left(\frac{\mu_0 I}{2\pi a}\hat{\theta}\right) \\ &= -\frac{IV_0}{2\pi a L}\hat{r} \end{aligned}$$

Note that the Poynting vector is pointing into the wire. This indicates that energy is flowing into the wire and being dissipated by the resistance.

(d) The power associated with the Poynting vector is given by its integral over an area, $P = \iint \vec{S} \cdot d\vec{A}$. To find the power going into the wire segment, we integrating over its surface and use the normal vector pointing into the volume of the wire. This gives,

$$\begin{aligned} P &= \iint \vec{S} d\vec{A} \\ &= \int_0^L \int_0^{2\pi} \left(-\frac{IV_0}{2\pi a L}\hat{r}\right) \cdot (-\hat{r} a d\theta dl) \\ &= \left(\frac{IV_0}{2\pi a L}\right)(2\pi a L) \\ P &= IV_0 \end{aligned}$$

Solution 7

Consider the following set of electric and magnetic field which correspond to a travelling plane wave:

$$\begin{aligned} E_x &= E_0 \sin(kz - \omega t) & B_x &= 0 \\ E_y &= 0 & B_y &= \left(\tfrac{E_0}{c}\right) \sin(kz - \omega t) \\ E_z &= 0 & B_z &= 0 \end{aligned}$$

(a) Computing the terms which show up in Maxwell's equation, we find,

$$\begin{aligned} \nabla \cdot \vec{E} &= 0 \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= \frac{\partial E_x}{\partial z}\hat{y} = kE_0 \cos(kz - \omega t)\,\hat{y} \\ \nabla \times \vec{B} &= -\frac{\partial B_y}{\partial z}\hat{x} = -k\left(\frac{E_0}{c}\right)\cos(kz - \omega t)\,\hat{x} \\ \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} &= -\frac{\omega}{c^2}E_0 \cos(kz - \omega t)\,\hat{x} \\ -\frac{\partial \vec{B}}{\partial t} &= \omega\left(\frac{E_0}{c}\right)\cos(kz - \omega t)\,\hat{y} \end{aligned}$$

The pair of Maxwell's equations related to the divergences are always satisfied by being equal to zero. The other two equations involving the curl are satisfied if $ck = \omega$. This is the usual relationship between frequency, wavelength, and the speed of the wave.

(b) Consider just the E_x component of the wave. If we fix $A = kz - \omega t$ for some constant A, then the electric field has some particular value. This relationship between z and t can be rewritten as $kz = \omega t + A$ or more transparently $z = \frac{\omega}{k}t + \frac{A}{k}$. Thus, at later times this particular value of the electric field is found at increasingly large values of z. As such, the wave can be said to be travelling in the $+\hat{z}$ direction with a speed $\frac{\omega}{k}$ which is unsurprisingly equal to the speed of light, c.

(c) The Poynting vector is defined as $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}$. Thus,

$$\begin{aligned} \vec{S} &= \frac{\vec{E}\times\vec{B}}{\mu_0} \\ &= \frac{1}{\mu_0}\left(E_0 \sin(kz - \omega t)\,\hat{x}\right) \times \left(\left(\frac{E_0}{c}\right)\sin(kz - \omega t)\,\hat{y}\right) \\ &= \sqrt{\frac{\epsilon_0}{\mu_0}}E_0^2 \sin^2(kz - \omega t)\,\hat{z} \end{aligned}$$

Thus, the Poynting vector is aligned with the direction the wave travels.

(d) The power associated with the Poynting vector is given by its integral over an area, $P = \iint \vec{S} \cdot d\vec{A}$. Integrating over the loop,

$$\begin{aligned} P &= \iint \vec{S} \cdot d\vec{A} \\ &= \iint \left(\sqrt{\frac{\epsilon_0}{\mu_0}}E_0^2 \sin^2(kz - \omega t)\,\hat{z}\right) \cdot (dA\,\hat{z}) \\ &= \sqrt{\frac{\epsilon_0}{\mu_0}}E_0^2 \pi R^2 \sin^2(\omega t) \end{aligned}$$

Averaging over a full cycle,

$$\langle P \rangle = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}E_0^2 \pi R^2$$

Solution 8

1. The direction $\hat{n} = (4/5, -3/5, 0)$; the wavelength is $\lambda = a/5 = 500$ nm; the frequency of the incident sunshine $f = c/\lambda = 6 \cdot 10^{14}$ Hz; and $B_x = 3/2$, thus $\vec{B} = 2.5 \cdot 10^{-6}(3/5, 4/5, 0)\cos(\omega t - k_x x + k_y y)$ webers/m^2.
2. The incident E-field = $-c\hat{k} \times \vec{B} = 7.5 \times 10^2(0, 0, -1)\cos(\omega t - k_x x + k_y y)$ Newtons/Coulomb
3. The Poynting vector of the incident light:

$$\vec{S} = 1.5 \times 10^3 \cos^2(\omega t - k_x x + k_y y)\hat{n}\ \mathrm{Watts/m^2}$$

$$<cos^2(\omega t - k_x x + k_y y)> = 0.5$$

$$\text{and so} \quad <S> = 7.5 \times 10^2\ \mathrm{Watts/m^2}$$

4. The reflected E-field = $7.5 \times 10^2(0, 0, 1)\cos(\omega t - k_x x - k_y y)$ Newtons/Coulomb
5. The total E-field = $1.5 \times 10^3(0, 0, 1)\sin(\omega t - k_x x)\sin(k_y y)$ Newtons/Coulomb, i.e. a propagating wave in the x direction and a standing wave in the y direction;
6. The additional energy per second absorbed by the Earth is

$$\vec{S} \cdot \vec{A} = 4.5 \times 10^{13}\ \mathrm{Watts}$$

which is much bigger than the $\sim 2 \times 10^{12}$ Watts of the average electrical power consumption of the world (~ 10 billion people world-wide times 100 Watts/person, or see e.g. Orders of magnitude (power)), so global warming will accelerate faster and faster, as more and more ice melts.

Solution 9

(a) We have the fields defined as

$$\mathbf{E}_1 = E_0 \hat{z} \sin(\omega t - k_x x - k_y y)$$
$$\mathbf{B}_1 = \mathbf{b}_1 \sin(\omega t - k_x x - k_y y)$$

where let us define, $\mathbf{b}_1 = (b_1^x, b_1^y, 0)$. Note that we can set $b_1^z = 0$ since for electromagnetic waves the magnetic and electric fields must be perperndicular. Using Maxwell's equations, we have $\nabla \times \mathbf{B}_1 = \frac{1}{c^2}\frac{\partial \mathbf{E}_1}{\partial t}$

$$\frac{\partial B_1^y}{\partial x} - \frac{\partial B_1^x}{\partial y} = \frac{\omega}{c^2} E_0 \cos(\omega t - k_x x - k_y y)$$
$$\Rightarrow -b_1^y k_x + b_1^x k_y = \frac{\omega}{c^2} E_0 \tag{1}$$

and using $\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t}$

$$\frac{\partial E_1^z}{\partial y} = -\frac{\partial B_1^x}{\partial t}$$
$$-\frac{\partial E_1^z}{\partial x} = -\frac{\partial B_1^y}{\partial t}$$
$$-k_y E_0 = -b_1^x \omega . \text{and} k_x E_0 = -b_1^y \omega \tag{2}$$

Therefore we have $(b_1^x, b_1^y) \equiv \left(\frac{k_y}{\omega} E_0, -\frac{k_x}{\omega} E_0\right)$Substituting back (2) into (1), we get,

$$k^2 = k_x^2 + k_y^2 = \frac{\omega^2}{c^2}$$
$$\Rightarrow \omega^2 = k^2 c^2 \tag{3}$$

The above fields are plane waves since the waves have a constant phase at a particular time along the plane $k_x x + k_y y = \text{const}$. Since we have an electromagnetic wave, it travels with a speed of light, c. The frequency, ν, and the wavelength, λ, is

$$\nu = \frac{\omega}{2\pi} = \frac{c\sqrt{k_x^2 + k_y^2}}{2\pi}$$
$$\lambda = \frac{2\pi}{k}$$

Finally, the direction of propagation can be obtained from the Poynting vector, $\mathbf{S}$,

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$
$$= \frac{1}{\mu_0}\frac{E_0^2}{\omega}(\hat{z} \times (k_y \hat{x} - k_x \hat{y}))\left[\sin(\omega t - k_x x - k_y y)\right]^2$$
$$= \frac{1}{\mu_0}\frac{E_0^2}{\omega}\sin^2(\omega t - k_x x - k_y y)(k_x \hat{x} + k_y \hat{y})$$

So, the wave is propagating along $\mathbf{k} = k_x \hat{x} + k_y \hat{y}$.

(b) Analogous to part (a), we make a replacement $k_y \to -k_y$. Thus required $\mathbf{b}_2 \equiv (b_1^x, b_1^y) \equiv \left(-\frac{k_y}{\omega} E_0, -\frac{k_x}{\omega} E_0\right)$.

(c) Adding the fields, we have

$$\begin{aligned}
\mathbf{E} &= \mathbf{E_1} + \mathbf{E_2} \\
&= E_0\hat{z}\left(\sin\left(\omega t - k_x x - k_y y\right) + \sin\left(\omega t - k_x x + k_y y\right)\right) \\
&= 2E_0\hat{z}\sin\left(\omega t - k_x x\right)\cos\left(k_y y\right) \\
\text{and } \mathbf{B} &= \mathbf{B_1} + \mathbf{B_2} \\
&= -\tfrac{2E_0}{\omega}\left(\hat{x}k_y\cos\left(\omega t - k_x x\right)\sin\left(k_y y\right) + \hat{y}k_x\sin\left(\omega t - k_x x\right)\cos\left(k_y y\right)\right)
\end{aligned}$$

Now we apply boundary conditions, $E_\parallel = 0$, at $y = y_1$ and $y = y_2$. The parallel component is along $\hat{z}$. So,

$$\cos\left(k_y y\right) = 0$$
$$\Rightarrow y_{1,2} = \left(2n_{1,2} + 1\right)\frac{\pi}{2k_y}$$

(d) $\mathbf{B}_\perp$ on the surface at $y = y_{1,2}$, is along $\hat{y}$. Again the $\hat{y}$ component of $\mathbf{B} \propto \cos\left(k_y y\right)$, and so vanishes at $y = y_{1,2}$.

(e) Setting $y_2 - y_1 = a$, we have

$$2\left(n_2 - n_1\right)\frac{\pi}{2} = k_y a$$

Also, we have $\omega^2 = c^2\left(k_x^2 + k_y^2\right)$, which means,

$$\begin{aligned}
k_x &= \sqrt{\frac{\omega^2}{c^2} - k_y^2} \\
&= \sqrt{\frac{\omega^2}{c^2} - \left(n_2 - n_1\right)^2\frac{\pi^2}{a^2}}
\end{aligned}$$

For k_x to be real, we therefore require, $\omega > c\left(n_2 - n_1\right)\frac{\pi}{a}$.

(f) The $\mathbf{E}$ and $\mathbf{B}$ fields move in the x direction because of the factor $(\omega t - k_x x)$ appearing in their expansion and the phase velocity is $v = \frac{\omega}{k_x}$. Alternatively, the time averaged -Poynting vector gives us the direction of the flow of energy, and hence the fields,

$$\mathbf{S} \propto \mathbf{E} \times \mathbf{B} \propto \left(\hat{y}\frac{k_y}{4}\sin\left(2\left(\omega t - k_x x\right)\right)\sin\left(2k_y y\right) + \hat{x}k_x\sin^2\left(\omega t - k_x x\right)\cos^2\left(k_y y\right)\right)$$

Upon time-averaging the $\hat{y}$-component vanishes and we have,

$$\langle \mathbf{S}_y \rangle = 0$$

Therefore, the wave travels only in the $\hat{x}$ direction.