<u>**Quiz Review 2**</u>

Relativity of EM Fields

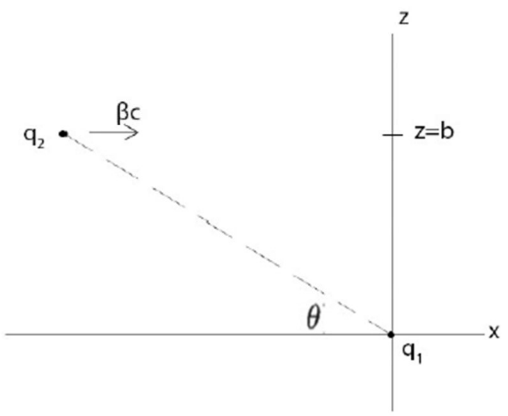
Problem 1

What is the magnetic field of a point charge moving with constant velocity? The electric field of a moving charge is given by

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q(1 - v_0^2/c^2)}{[1 - (v_0^2/c^2)\sin^2\theta]^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}.$$

Problem 2

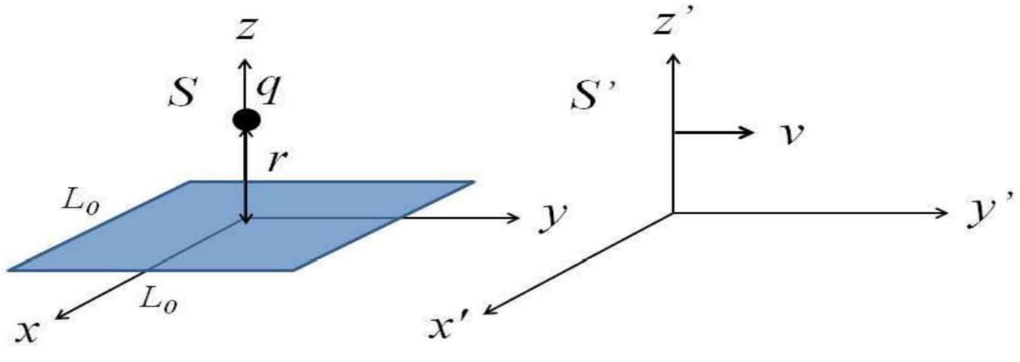
A charge q_1 is at rest in the origin, and a charge q_2 moves with speed βc in the x direction, along the line *z=b*. For what angle Θ shown in the figure will the horizontal component of the force on q_1 be maximum? What is Θ in the $\beta \approx 1$ and $\beta \approx 0$ limits?



Problem 3

A very large flat plate, which has sides $L_0 \times L_0$ as measured in its rest frame S, has charge Q uniformly distributed over its surface. Thus the surface charge density is $\sigma = Q / L_0^2$. A test point charge q is located at a perpendicular distance, $r \ll L_0$, above the plate and is stationary in S.

The purpose of this problem is to showcase two ways to solve a problem, either by exclusively considering fields (parts a and b) or by explicitly considering the sources (parts c and d). Along the way we also wish you to appreciate the power of vector algebra.



(a) Let $\vec{E}$ and $\vec{B}$ be the fields in the vicinity of the test charge in frame S. Using the relativistic transformation of fields, compute the corresponding fields $\vec{E}'$ and $\vec{B}'$ in frame S', which is moving along the y-axis with a speed of v, in terms of the non-zero components of $\vec{E}$ and $\vec{B}$. [1pt]

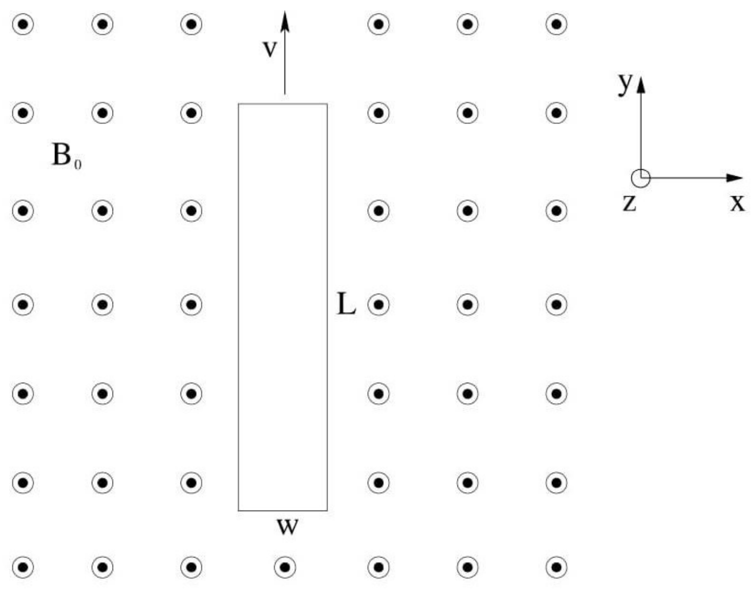
(b) Using the expressions for $\vec{E}'$ and $\vec{B}'$ found in part (a), compute the net force $\vec{F}'$ on the test charge in S'. Transform this force back to frame S to find the force $\vec{F}$ in its rest frame. [1pt] Note the helpful vector identity $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$

(c) Compute $\vec{E}$ and $\vec{B}$ at the position of the test charge in terms of q, Q and L_0 by explicitly considering the electrical charges and currents in frame S. [1pt]

(d) Compute $\vec{E}'$ and $\vec{B}'$ at the position of the test charge in terms of q, Q and L_0 by explicitly considering the electrical charges and currents as seen in frame S'. Verify that these values agree with those computed in (a). [2pts]

Problem 4

In the lab (S), a neutral block of metal moves in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ as shown below. It moves with a constant velocity $\mathbf{v} = v\hat{\mathbf{y}}$, and its width w is much smaller than its length (L) and height. A surface charge builds up on the block's left and right faces; you can assume it is evenly distributed.



(a) We assume to a good approximation that the electric field $\mathbf{E}$ far from the block is zero in the lab frame. Find the electric field $\mathbf{E}'$ far from the block in the block's rest frame S'.

(b) Find the surface charge densities σ'_L and σ'_R on the left and right faces of the block, as measured in the rest frame.

(c) By transforming σ'_L and σ'_R, find the potential difference $\Delta\phi$ between the faces of the block in the lab frame. What is $\Delta\phi'$, the potential difference in the block's rest frame?

[Note that potential as a line integral of $\mathbf{E}$ still makes sense as long as $\partial\mathbf{B}/\partial t = 0$, at least in the region of interest.]

Solution 1

In the particle's *rest* frame the magnetic field is zero (everywhere), so in a system moving with velocity $-\mathbf{v}$ (in which the *particle* is moving at velocity $+\mathbf{v}$)

$$\mathbf{B} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}).$$

The magnetic field, then, is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{qv(1 - v^2/c^2)\sin\theta}{[1 - (v^2/c^2)\sin^2\theta]^{3/2}} \frac{\hat{\boldsymbol{\phi}}}{R^2},$$

where $\hat{\boldsymbol{\phi}}$ aims counterclockwise as you face the oncoming charge. Incidentally, in the nonrelativistic limit ($v^2 \ll c^2$), reduces to

$$\mathbf{B} \approx \frac{\mu_0}{4\pi} q \frac{\mathbf{v} \times \hat{\mathbf{R}}}{R^2},$$

which is exactly what you would get by naïve application of the Biot-Savart law to a point charge

Solution 2

The magnitude of the electric field is given in Eq. (5.15), with $r' = b/\sin\theta$. We are concerned with the horizontal component, so this brings in a factor of $\cos\theta$. We therefore want to maximize the function

$$E_x \propto \frac{\sin^2\theta\cos\theta}{(1 - \beta^2\sin^2\theta)^{3/2}}. \qquad (12.245)$$

Setting the derivative equal to zero and simplifying yields

$$(1-\beta^2\sin^2\theta)(2\cos^2\theta-\sin^2\theta)+3\beta^2\sin^2\theta\cos^2\theta=0. \quad (12.246)$$

Using $\cos^2\theta = 1-\sin^2\theta$ and solving for $\sin^2\theta$ gives

$$\sin\theta = \sqrt{\frac{2}{3-\beta^2}}. \quad (12.247)$$

If $\beta \approx 1$ then $\theta \approx 90°$, which is reasonable. The largeness of the field near $90°$ wins out over the smallness of the $\cos\theta$ factor involved in taking the horizontal component. However, if $\theta = 90°$ exactly, then the horizontal force is zero.

If $\beta \approx 0$ then $\sin\theta \approx \sqrt{2/3} \implies \theta \approx 54.7°$ (or $125.3°$). You can quickly check from scratch that this is the correct result in the nonrelativistic case, where the horizontal component of the force is proportional to $\sin^2\theta\cos\theta$.

Solution 3

a

In the lab frame, there is no moving charges and so $\vec{B}=0$. The electrical force is due to the electrostatic field of the charged plate. For $r \ll L_0$ we argue that field is primarily along the z axis. Thus

$$\vec{E} = E\hat{z} = \vec{E}_\perp, \qquad \vec{B} \quad (1)$$

where the usage of $\perp$ anticipates calculations in frame S'. Though we will not need it for this section we will note that

$$\vec{E}_\perp = \frac{\sigma}{2\epsilon_0}. \quad (2)$$

Thus the net force on the test particle is

$$\vec{F} = q\vec{E}_\perp. \quad (3)$$

Frame S' is moving with $\vec{v}=v\hat{x}$ with respect to frame S. There are no longitudinal fields in frame S and thus there are no longitudinal forces in frame S' also. We apply the Lorentz transformation for the transverse components and find

$$\vec{E}' = \gamma\vec{E}_\perp \quad (4)$$
$$\vec{B}' = -\frac{\gamma}{c^2}\vec{v}\times\vec{E}_\perp. \quad (5)$$

Here, as usual, $\gamma = (1-\beta^2)^{-1/2}$ where $\beta = v/c$.

b

In frame S' the test particle is moving with velocity $\vec{u} = \vec{v}$. Thus the electrical and magnetic force experienced by the particle is, respectively,

$$F'_E = \gamma q \vec{E}_\perp \tag{6}$$

$$F'_B = q\vec{u} \times B' = -\gamma \frac{q}{c^2} \vec{u} \times (\vec{v} \times \vec{E}_\perp). \tag{7}$$

Noting the vector identity $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ we simplify the last equation and find

$$F'_B = -\gamma q \frac{v^2}{c^2} \vec{E}_\perp. \tag{8}$$

The sum of the forces (Equations 6 and 8) is

$$F'_B = q\gamma \left(1 - \frac{v^2}{c^2}\right) \vec{E}_\perp = \frac{1}{\gamma} q \vec{E}_\perp \tag{9}$$

In the lab frame the test particle is at rest. Thus the corresponding force (which is entirely perpendicular) is larger in the lab frame by γ and is

$$\vec{F} = q\vec{E}_\perp. \tag{10}$$

which should be compared to Equation 3.

c

In the lab frame the charge density of the plate is $\sigma = Q/L_0$. There are no moving charges and so $\vec{B} = 0$. In the approximation of $r \ll L_0$ we have

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{z}. \tag{11}$$

d

In frame S' the charge density on the plate is larger by γ owing to Lorentz contraction (along the y-axis). Also the charge plate is now moving with velocity $-\vec{v}$. Thus we expect to see both electrostatic and magnetic fields. The moving charge plate appears as a current sheet with a surface current, $\vec{\mathcal{J}}$. The surface current is the product of the charge density and the velocity of the plate, or $\vec{\mathcal{J}} = -\gamma\sigma\vec{v}$.

The electrostatic field at the location of the test particle, $\vec{r} = (0, 0, r)$ is

$$\vec{E}' = \frac{\gamma\sigma}{2\epsilon_0} \hat{z} = \gamma\vec{E}. \tag{12}$$

We derived same result earlier (Equation 4). The current sheet produces a magnetic field which has to be perpendicular to the current sheet (Biot-Savart law). For $r \ll L_0$ (or even L_0/γ) we can assume that the magnetic field is close to that produced by a moving plate of infinite size. Next, the application of the right hand rule informs us that for $z > 0$ the field is directed along x-axis (and the opposite for $z < 0$). Application of Ampere?s law (with a rectangular loop passing through q and in the x-z plane) yields the field at $\vec{r} = r\hat{z}$ as

$$\vec{B}? = \frac{\mu_0 \vec{\mathcal{J}}}{2} \times \hat{z} = -\frac{1}{2}\gamma\mu_0\sigma v\hat{x}. \tag{13}$$

Noting $\epsilon_0\mu_0 = c^{-2}$ we simplify the above equation and find

$$\vec{B}' = -\gamma \frac{v}{c^2} \frac{\sigma}{2\epsilon_0} \hat{x}. \tag{14}$$

Solution 4

(a)

In the rest frame, far from the block, there will be a uniform electric field pointing to the right. To calculate it, note first that the block's rest frame S' moves with speed $\mathbf{v}_r = v\hat{\mathbf{y}}$ in (with respect to) frame S. There's no $E_{//}$ in either frame. The new $\mathbf{E}'_\perp$ is

$$\mathbf{E}_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v}_r \times \mathbf{B}_\perp) \tag{1}$$

$$= \gamma\ \mathbf{v}_r \times \mathbf{B}_\perp \tag{2}$$

$$= \gamma\, \mathbf{v}_r B_0 \hat{\mathbf{x}}. \tag{3}$$

(b)

The electric field calculated in part (a) is the "background" field around the block as well. Since the block is a conductor (at rest!), its charges will rearrange themselves to cancel the field inside; in other words, they will produce a field inside the block uniform and *opposite* to $\mathbf{E}'$.

Using what you know about the field of a charged plane, or the fact that the discontinuity in electric field at a charged surface must be σ/ε_0, we've got

$$\sigma'_{L,R} = \mp E' \varepsilon_0 \quad = \mp\gamma \tag{4}$$

(c)

The charge densities get Lorentz-contracted in the lab frame, so

$$\sigma_{L,R} = \gamma\sigma_{L,R} = \mp\gamma^2\, \mathbf{v}_r\, \varepsilon_0 B_0 \tag{5}$$

This corresponds to a uniform electric field inside the block

$$\mathbf{E} = -\gamma^2\, \mathbf{v}_r\, B_0 \tag{6}$$

and so

$$\Delta\phi = wE = \gamma^2\, \mathbf{v}_r\, wB_0. \tag{7}$$

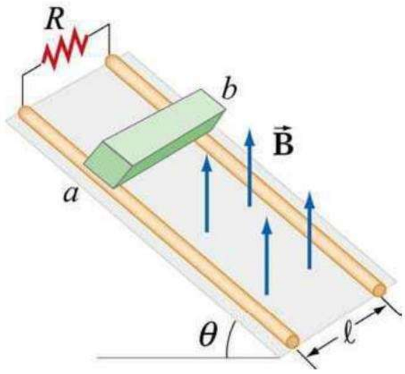
In the rest frame, there is *no net electric field* inside the block, so

$$\Delta\phi' = 0. \tag{8}$$

Faraday's Law

Problem 5

A conducting bar of mass m slides down two frictionless conducting rails which make an angle θ with the horizontal, separated by a distance ℓ and connected at the top by a resistor R, as shown in the figure. In addition, a uniform magnetic field $\vec{\mathbf{B}}$ is applied vertically upward. The bar is released from rest and slides down. At time t the bar is moving along the rails at speed $v(t)$.



(a) Find the induced current in the bar at time t. Which way does the current flow, from a to b or b to a?

(b) Find the terminal speed v_T of the bar.

After the terminal speed has been reached,

(c) What is the induced current in the bar?

(d) What is the rate at which electrical energy is being dissipated through the resistor?

(e) What is the rate of work done by gravity on the bar? The rate at which work is done is $\vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$. How does this compare to your answer in (d)? Why?

Solutions :

(a) The flux between the resistor and bar is given by

$$\Phi_B = B \ell x(t) \cos\theta$$

where $x(t)$ is the distance of the bar from the top of the rails.

Then,

$$\varepsilon = -\frac{d}{dt}\Phi_B = -\frac{d}{dt} B \ell x(t) \cos\theta = -B \ell v(t) \cos\theta$$

Because the resistance of the circuit is R, the magnitude of the induced current is

$$I = \frac{|\varepsilon|}{R} = \frac{B \ell v(t) \cos\theta}{R}$$

By Lenz's law, the induced current produces magnetic fields which tend to oppose the change in magnetic flux. Therefore, the current flows clockwise, from b to a across the bar.

(b) At terminal velocity, the net force along the rail is zero, that is gravity is balanced by the magnetic force:

$$mg\sin\theta = IB\ell\cos\theta = \left(\frac{B\ell v_t(t)\cos\theta}{R}\right)B\ell\cos\theta$$

or

$$v_t(t) = \frac{Rmg\sin\theta}{(B\ell\cos\theta)^2}$$

(c)

$$I = \frac{B\ell v_t(t)\cos\theta}{R} = \frac{B\ell\cos\theta}{R}\left(\frac{Rmg\sin\theta}{(B\ell\cos\theta)^2}\right) = \frac{mg\sin\theta}{B\ell\cos\theta} = \frac{mg}{B\ell}\tan\theta$$

(d) The power dissipated in the resistor is

$$P = I^2R = \left(\frac{mg}{B\ell}\tan\theta\right)^2 R$$

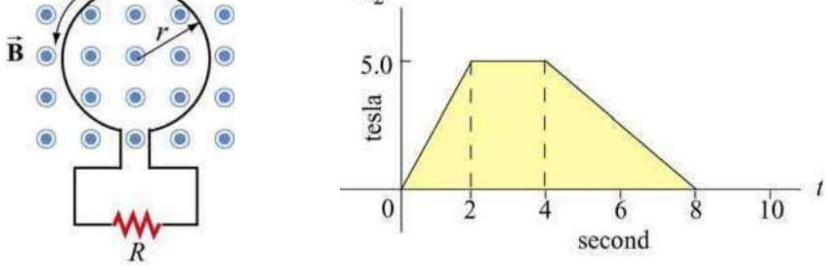
(e)

$$\vec{F}\cdot\vec{V} = (mg\sin\theta)v_t(t) = mg\sin\theta\left(\frac{Rmg\sin\theta}{(B\ell\cos\theta)^2}\right) = \left(\frac{mg}{B\ell}\tan\theta\right)^2 R = P$$

That is, they are equal. All of the work done by gravity is dissipated in the resistor, which is why the rod isn't accelerating past its terminal velocity.

Problem 6

A uniform magnetic field $\vec{\mathbf{B}}$ is perpendicular to a one-turn circular loop of wire of negligible resistance, as shown in the figure below. The field changes with time as shown (the z direction is out of the page). The loop is of radius $r = 50$ cm and is connected in series with a resistor of resistance $R = 20\ \Omega$. The "+" direction around the circuit is indicated in the figure.



(a) What is the expression for EMF in this circuit in terms of $B_z(t)$ for this arrangement?

(b) Plot the EMF in the circuit as a function of time. Label the axes quantitatively (numbers and units). Watch the signs. Note that we have labeled the positive direction of the emf in the left sketch consistent with the assumption that positive $\vec{\mathbf{B}}$ is out of the paper.

(c) Plot the current *I* through the resistor *R*. Label the axes quantitatively (numbers and units). Indicate with arrows on the sketch the *direction* of the current through *R* during each time interval.

(d) Plot the power dissipated in the resistor as a function of time.

Solutions:

(a) When we choose a "+" direction around the circuit shown in the figure above, then we are also specifying that magnetic flux out of the page is positive. (The unit vector $\hat{\mathbf{n}} = +\hat{\mathbf{k}}$ points out of the page). Thus the dot product becomes

$$\vec{\mathbf{B}} \cdot \hat{\mathbf{n}} = \vec{\mathbf{B}} \cdot \hat{\mathbf{k}} = B_z . \qquad (0.1)$$

From the graph, the z-component of the magnetic field B_z is given by

$$B_z = \begin{cases} (2.5\ \mathrm{T \cdot s^{-1}})t\,;\ 0 < t < 2\ \mathrm{s} \\ 5.0\ \mathrm{T}\,;\ 2\ \mathrm{s} < t < 4\ \mathrm{s} \\ 10\ \mathrm{T} - (1.25\ \mathrm{T \cdot s^{-1}})t\,;\ 4\ \mathrm{s} < t < 8\ \mathrm{s} \\ 0\,;\ t > 8\ \mathrm{s} \end{cases} . (0.2)$$

The derivative of the magnetic field is then

$$\frac{dB_z}{dt} = \begin{cases} 2.5\ \mathrm{T \cdot s^{-1}}\,;\ 0 < t < 2\ \mathrm{s} \\ 0\,;\ 2\ \mathrm{s} < t < 4\ \mathrm{s} \\ -1.25\ \mathrm{T \cdot s^{-1}}\,;\ 4\ \mathrm{s} < t < 8\ \mathrm{s} \\ 0\,;\ t > 8\ \mathrm{s} \end{cases} . \qquad (0.3)$$

The magnetic flux is therefore

$$\Phi_{magnetic} = \iint \vec{\mathbf{B}} \cdot \hat{\mathbf{n}}\, d\vec{\mathbf{A}} = \iint B_z dA = B_z \pi r^2 . (0.4)$$

The electromotive force is

$$\mathcal{E} = -\frac{d}{dt}\Phi_{magnetic} = -\frac{dB_z}{dt}\pi r^2 . \qquad (0.5)$$

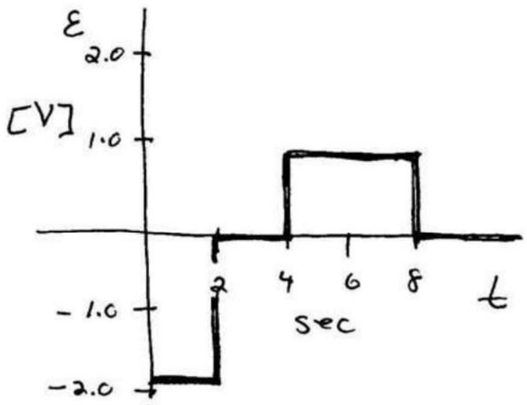
So we calculate the electromotive force by substituting Eq. (0.3) into Eq. (0.5) yielding

$$\mathcal{E}=\begin{cases}-(2.5\,\mathrm{T\cdot s^{-1}})\pi r^2\,;0<t<2\,\mathrm{s}\\0\,;2\,\mathrm{s}<t<4\,\mathrm{s}\\(1.25\,\mathrm{T\cdot s^{-1}})\pi r^2\,;4\,\mathrm{s}<t<8\,\mathrm{s}\\0\,;t>8\,\mathrm{s}\end{cases}. \qquad (0.6)$$

Using $r=0.5\,\mathrm{m}$, the electromotive force is then

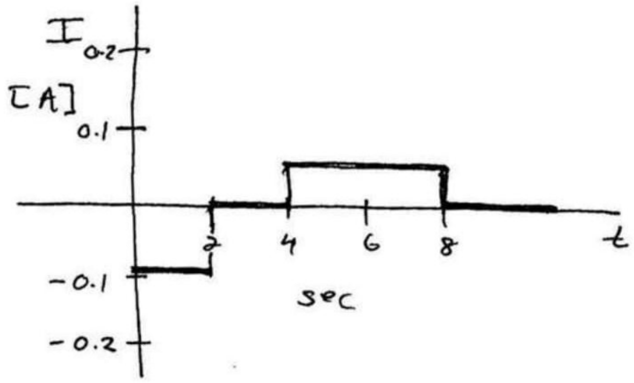
$$\mathcal{E}=\begin{cases}-1.96\,\mathrm{V}\,;0<t<2\,\mathrm{s}\\0\,;2\,\mathrm{s}<t<4\,\mathrm{s}\\0.98\,\mathrm{V}\,;4\,\mathrm{s}<t<8\,\mathrm{s}\\0\,;t>8\,\mathrm{s}\end{cases} \qquad (0.7)$$

(b)



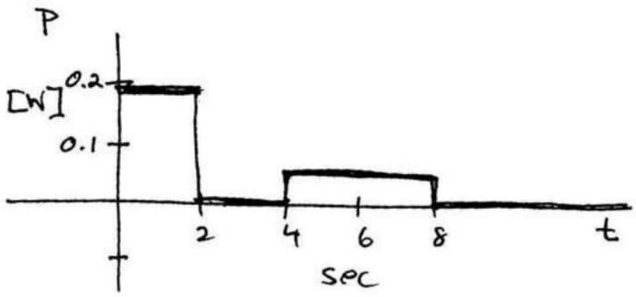
(c) The current through the resistor ($R=20\,\Omega$) is given by

$$I=\frac{\mathcal{E}}{R}=\begin{cases}-9.8\times10^{-2}\,\mathrm{A}\,;0<t<2\,\mathrm{s}\\0\,;2\,\mathrm{s}<t<4\,\mathrm{s}\\4.9\times10^{-2}\,\mathrm{A}\,;4\,\mathrm{s}<t<8\,\mathrm{s}\\0\,;t>8\,\mathrm{s}\end{cases}. \qquad (0.8)$$



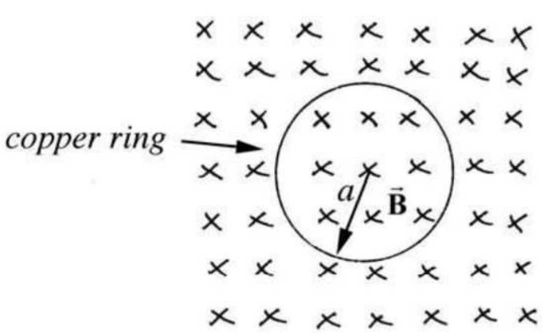
(d) The power dissipated in the resistor is given by

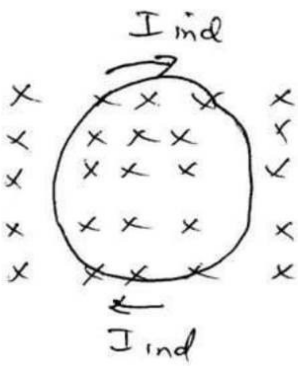
$$P = I^2 R = \begin{cases} 1.9\times10^{-1}\ \text{W}\,;\ 0 < t < 2\ \text{s} \\ 0\,;\ 2\ \text{s} < t < 4\ \text{s} \\ 4.8\times10^{-2}\ \text{W}\,;\ 4\ \text{s} < t < 8\ \text{s} \\ 0\,;\ t > 8\ \text{s} \end{cases}. \quad (0.9)$$



Problem 7

Consider a copper ring of radius a and resistance R. The loop is in a constant magnetic field $\vec{\mathbf{B}}$ of magnitude B_0 perpendicular to the plane of the ring (pointing into the page, as shown in the diagram).





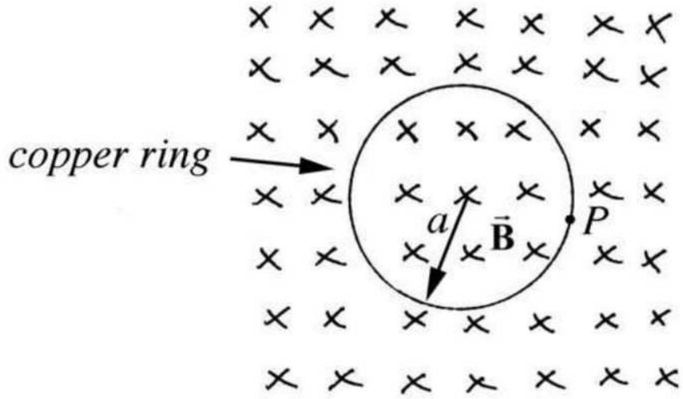
(a) What is the magnetic flux Φ through the ring? Express your answer in terms of B_0, a, R, and μ_0 as needed.

Now, the magnitude of the magnetic field is decreased during a time interval from $t = 0$ to $t = T$ according to

$$B(t) = B_0\left(1 - \frac{t}{T}\right), \text{ for } 0 < t \le T$$

(b) What are the magnitude and direction (draw the direction on the figure above) of the current I in the ring? Express your answer in terms of B_0, T, a, R, t, and μ_0 as needed.

(c) What is the total charge Q that has moved past a fixed point P in the ring during the time interval that the magnetic field is changing? Express your answer in terms of B_0, T, a, R, t, and μ_0 as needed.



Solutions:

(a) $\Phi = B_0 \pi a^2$

(b) The external flux is into the page and decreasing so the induced current is in the clockwise direction producing flux into the page through the ring opposing the change. The magnitude of the induced current is non-zero during the interval $0 < t \le T$ and is equal to

$$I = \frac{1}{R}\left|\frac{d\Phi}{dt}\right| = \frac{1}{R}\left|\frac{d}{dt}\left(B_0\left(1-\frac{t}{T}\right)\pi a^2\right)\right| = \frac{1}{R}\left|\frac{d}{dt}\left(B_0\left(1-\frac{t}{T}\right)\pi a^2\right)\right| = \frac{B_0 \pi a^2}{TR}, \textbf{ for } 0 < t \le T$$

(c) The total charge moving past a fixed point P in the ring is the integral

$$Q = \int_0^T I dt = \int_0^T \frac{B_0 \pi a^2}{TR} dt = \frac{B_0 \pi a^2}{R}.$$