Lorentz Force

Problem 1

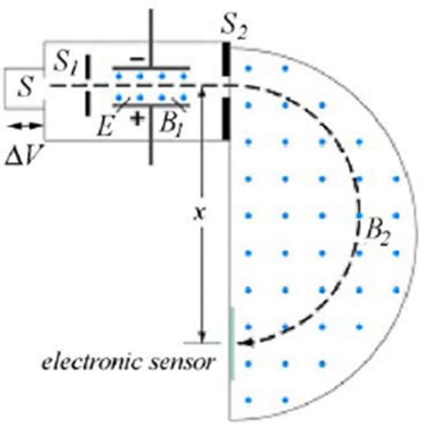
A copper wire of diameter d carries a current density $\vec{\mathbf{J}}$ at the earth's equator where the earth's magnetic field is horizontal, points north, and has magnitude $\left|\vec{\mathbf{B}}_{earth}\right| = 0.5 \times 10^{-4}$ T. The wire lies in a plane that is parallel to the surface of the earth and is oriented in the east-west direction. The density of copper is $\rho_{Cu} = 8.9 \times 10^{3}$ kg $\cdot$ m^{-3}. The resistivity of copper is $\rho_r = 1.7 \times 10^{-8} \Omega \cdot \text{m}$.

a) How large must $\vec{\mathbf{J}}$ be, and which direction must it flow in order to levitate the wire? Use $g = 9.8 \text{m} \cdot \text{s}^{-2}$

b) When the wire is floating how much power will be dissipated per cubic centimeter?

Problem 2

Shown below are the essentials of a commercial mass spectrometer. This device is used to measure the composition of gas samples, by measuring the abundance of species of different masses. An ion of mass m and charge $q = +e$ is produced in source S, a chamber in which a gas discharge is taking place. The initially stationary ion leaves S, is accelerated by a potential difference $\Delta V > 0$, and then enters a selector chamber, S_1, in which there is an adjustable magnetic field $\vec{\mathbf{B}}_1$, pointing out of the page and a deflecting electric field $\vec{\mathbf{E}}$, pointing from positive to negative plate. Only particles of a uniform velocity $\vec{\mathbf{v}}$ leave the selector. The emerging particles at S_2, enter a second magnetic field $\vec{\mathbf{B}}_2$, also pointing out of the page. The particle then moves in a semicircle, striking an electronic sensor at a distance x from the entry slit. Express your answers to the questions below in terms of $E \equiv \left|\vec{\mathbf{E}}\right|$, e, x, m, $B_2 \equiv \left|\vec{\mathbf{B}}_2\right|$, and ΔV.



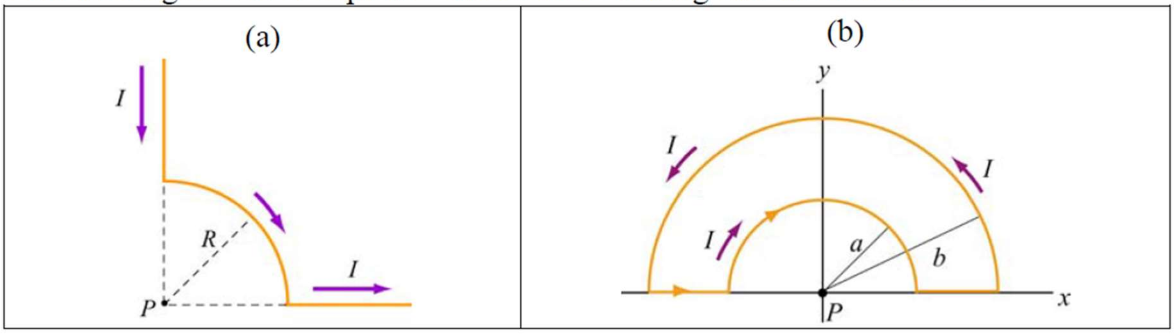
a) What magnetic field $\vec{\mathbf{B}}_1$ in the selector chamber is needed to insure that the particle travels straight through?

b) Find an expression for the mass of the particle after it has hit the electronic sensor at a distance x from the entry slit

Biot Savart's Law

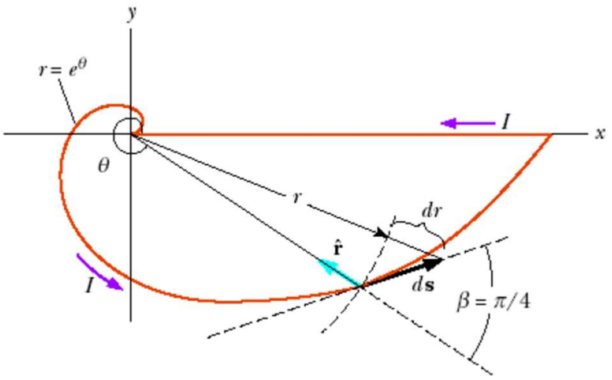
Problem 1

Find the magnetic field at point P due to the following current distributions:



Problem 2

A wire carrying a current I is bent into the shape of an exponential spiral, $r = e^{\theta}$, from $\theta =$ 0 to $\theta = 2\pi$ as shown in the figure below.



To complete a loop, the ends of the spiral are connected by a straight wire along the x axis. Find the magnitude and direction of $\vec{\mathbf{B}}$ at the origin.

Hint: Use the Biot–Savart law. The angle β between a radial line and its tangent line at any point on the curve $r = f(\theta)$ is related to the function in the following way:

$$\tan\beta = \frac{r}{dr/d\theta}$$

Thus in this case $r = e^{\theta}$, $\tan\beta = 1$ and $\beta = \pi/4$. Therefore, the angle between $d\vec{s}$ and $\hat{\mathbf{r}}$ is $\pi - \beta = 3\pi/4$. Also

$$ds = \frac{dr}{\sin(\pi/4)} = \sqrt{2}\, dr$$

Ampere's Law

Problem-Solving Strategy for Ampere's Law

Ampere's law states that the line integral of $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around any closed loop is proportional to the total steady current passing through any surface that is bounded by the closed loop:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}$$

To apply Ampere's law to calculate the magnetic field, we use the following procedure:

Step 1: Identify the 'symmetry' properties of the charge distribution.

Step 2: Determine the direction of the magnetic field

Step 3: Decide how many different spatial regions the current distribution determines

For each region of space…

Step 4: Choose an Amperian loop along each part of which the magnetic field is either constant or zero

Step 5: Calculate the current through the Amperian Loop

Step 6: Calculate the line integral $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around the closed loop.

Step 7: Equate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ with $\mu_0 I_{\text{enc}}$ and solve for $\vec{\mathbf{B}}$.

Problem 1

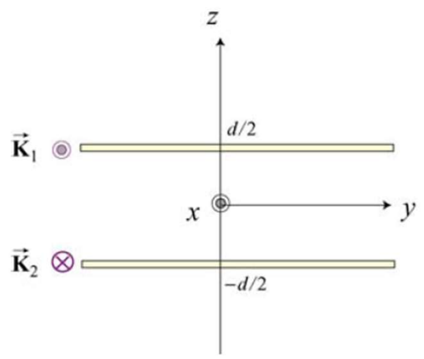
Consider an infinitely long, cylindrical conductor of radius R carrying a current I with a *non-uniform* current density $J = \alpha r^2$, where α is a constant and r is the distance from the center of the cylinder.

(a) Find the magnetic field everywhere.

(b) Plot the magnitude of the magnetic field as a function of r.

Problem 2

Consider two infinitely large sheets lying in the *xy*-plane separated by a distance *d* carrying surface current densities $\vec{\mathbf{K}}_1 = K\hat{\mathbf{i}}$ and $\vec{\mathbf{K}}_2 = -K\hat{\mathbf{i}}$ in the opposite directions, as shown in the figure below (The extent of the sheets in the *y* direction is infinite.) Note that *K* is the current per unit width perpendicular to the flow.



a) Find the magnetic field everywhere due to $\vec{\mathbf{K}}_1$.

b) Find the magnetic field everywhere due to $\vec{\mathbf{K}}_2$.

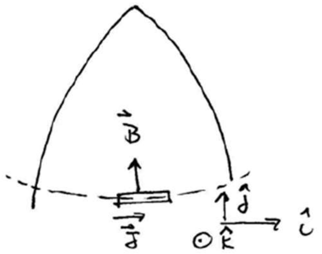
c) Applying superposition principle, find the magnetic field everywhere due to both current sheets.

d) How would your answer in (c) change if both currents were running in the same direction, with $\vec{\mathbf{K}}_1 = \vec{\mathbf{K}}_2 = K\hat{\mathbf{i}}$?

Lorentz Force

Solution 1

At the equator, the magnetic field is pointing north. Choose unit vectors such that $\hat{\mathbf{i}}$ points east, $\hat{\mathbf{j}}$ points north, and $\hat{\mathbf{k}}$ points up. Let $\vec{\mathbf{J}} = J_x\,\hat{\mathbf{i}}$ (with the sign of J_x to be determined), $\vec{\mathbf{B}}_{earth} = B_{earth}\,\hat{\mathbf{j}}$. .



Then the magnetic force $d\vec{\mathbf{F}}_{mag}$ on the a small volume of wire dV_{vol} is

$$d\vec{\mathbf{F}}_{mag} = \vec{\mathbf{J}}dV_{vol} \times \vec{\mathbf{B}}_{earth} = J_x dV_{vol}\,\hat{\mathbf{i}} \times B_{earth}\hat{\mathbf{j}} = J_x B_{earth} dV_{vol}\,\hat{\mathbf{k}}.$$

In order to balance the gravitational force this must point upwards hence $J_x > 0$; the current flows from west to east in the wire. The total force on the small element of the wire is zero so

$$\vec{\mathbf{0}} = d\vec{\mathbf{F}}_{grav} + d\vec{\mathbf{F}}_{mag} = \rho_{Cu} dV_{vol} g(-\hat{\mathbf{k}}) + J_x B_{earth} dV_{vol}\,\hat{\mathbf{k}}.$$

We can solve the above equation for J_x:

$$J_x = \frac{\rho_{Cu} g}{B_{earth}}$$

$$J_x = \frac{(8.9\times10^3\ \text{kg}\cdot\text{m}^{-3})(9.8\ \text{m}\cdot\text{s}^{-2})}{(0.5\times10^{-4}\ \text{T})} = 1.74\times10^9\ \text{A}\cdot\text{m}^{-2}$$

(b) Let $A = \pi(d/2)^2$ denote the cross-sectional area of the wire. The power dissipated per volume $dV_{vol} = Adl$ where dl is a unit length of wire is given by

$$\frac{P}{dV_{vol}} = \frac{I^2 R}{dV_{vol}}.$$

.

Let The current that flows in the wire is given by is given by $I = J_x A$. The resistance per unit length dl is given by $R = \rho_r dl / A$. So the above equation becomes

$$\frac{P}{dV_{vol}} = \frac{(J_x A)^2(\rho_r dl/A)}{Adl} = \rho_r J_x^{\,2}$$
$$= (1.7\times10^{-8}\ \Omega\cdot\text{m})(1.74\times10^9\ \text{A}\cdot\text{m}^{-2})^2.$$
$$= (5.2\times10^{10}\ \text{W}\cdot\text{m}^{-3})$$
$$\cong 50\ \text{kW}\cdot\text{cm}^{-3}$$

The wire will get very hot!

Solution 2

(a) We first find an expression for the speed of the particle after it is accelerated by the potential difference ΔV , in terms of m , e , and ΔV . The change in kinetic energy is $\Delta K = (1/2)mv^2$. The change in potential energy is $\Delta U = -e\Delta V$ From conservation of energy, $\Delta K = -\Delta U$, we have that

$$(1/2)mv^2 = e\Delta V .$$

So the speed is

$$v = \sqrt{\frac{2e\Delta V}{m}}$$

Inside the selector the force on the charge is given by

$$\vec{\mathbf{F}}_e = e(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_1) .$$

If the particle travels straight through the selector then force on the charge is zero, therefore

$$\vec{\mathbf{E}} = -\vec{\mathbf{v}} \times \vec{\mathbf{B}}_1 .$$

Since the velocity is to the right in the figure above (define this as the $+\hat{\mathbf{i}}$ direction), the electric field points up (define this as the $+\hat{\mathbf{j}}$ direction) from the positive plate to the negative plate, and the magnetic field is pointing out of the page (define this as the $+\hat{\mathbf{k}}$ direction) . Then

$$E\hat{\mathbf{j}} = -v\hat{\mathbf{i}} \times B_1\hat{\mathbf{k}} = vB_1\hat{\mathbf{j}} .$$

So

$$\vec{\mathbf{B}}_1 = \frac{E}{v}\hat{\mathbf{k}} = \sqrt{\frac{m}{2e\Delta V}}E\,\hat{\mathbf{k}}$$

(b) The force on the charge when it enters the magnetic field $\vec{\mathbf{B}}_2$ is given by

$$\vec{\mathbf{F}}_e = ev\hat{\mathbf{i}} \times B_2\hat{\mathbf{k}} = -evB_2\hat{\mathbf{j}} .$$

This force points downward and forces the charge to start circular motion. You can verify this because the magnetic field only changes the direction of the velocity of the particle and not its magnitude which is the condition for circular motion. When in circular motion the acceleration is towards the center. In particular when the particle just enters the field $\vec{\mathbf{B}}_2$, the acceleration is downward

$$\vec{\mathbf{a}} = -\frac{v^2}{x/2}\hat{\mathbf{j}} .$$

Newton's Second Law becomes

$$-evB_2\hat{\mathbf{j}} = -m\frac{v^2}{x/2}\hat{\mathbf{j}} .$$

Thus the particle hits the electronic sensor at a distance

$$x = \frac{2mv}{eB_2} = \frac{2}{eB_2}\sqrt{2e\Delta Vm}$$

from the entry slit. The mass of the particle is then

$$m = \frac{eB_2{}^2x^2}{8\Delta V} .$$

Biot Savart's Law

Solution 1

(a) The fields due to the straight wire segments are zero at P because $d\vec{\mathbf{s}}$ and $\hat{\mathbf{r}}$ are parallel or anti-parallel. For the field due to the arc segment, the magnitude of the magnetic field due to a differential current carrying element is given in this case by

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{R^2} = \frac{\mu_0}{4\pi}\frac{IRd\theta(\sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}})\times(-\cos\theta\,\hat{\mathbf{i}} - \sin\theta\,\hat{\mathbf{j}})}{R^2}$$
$$= -\frac{\mu_0}{4\pi}\frac{I(\sin^2\theta + \cos^2\theta)d\theta}{R}\hat{\mathbf{k}} = -\frac{\mu_0}{4\pi}\frac{Id\theta}{R}\hat{\mathbf{k}}$$

Therefore,

$$\vec{\mathbf{B}} = -\int_0^{\pi/2}\frac{\mu_0 I}{4\pi R}d\theta\,\hat{\mathbf{k}} = -\frac{\mu_0 I}{4\pi R}\left(\frac{\pi}{2}\right)\hat{\mathbf{k}} = -\left(\frac{\mu_0 I}{8R}\right)\hat{\mathbf{k}} \text{ (or, into the page).}$$

(b) There is no magnetic field due to the straight segments because point P is along the lines. Using the general expression for $d\vec{\mathbf{B}}$ obtained in (a), for the outer segment, we have

$$\vec{\mathbf{B}}_{\text{out}} = \int_0^{\pi}\frac{\mu_0}{4\pi}\frac{Id\theta}{b}\hat{\mathbf{k}} = \left(\frac{\mu_0 I}{4b}\right)\hat{\mathbf{k}}$$

Similarly, the contribution to the magnetic field from the inner segment is

$$\vec{\mathbf{B}}_{\text{in}} = \int_{\pi}^{0}\frac{\mu_0}{4\pi}\frac{Id\theta}{a}\hat{\mathbf{k}} = -\left(\frac{\mu_0 I}{4a}\right)\hat{\mathbf{k}}.$$

Therefore the net magnetic field at Point P is

$$\vec{\mathbf{B}}_{\text{net}} = \vec{\mathbf{B}}_{\text{out}} + \vec{\mathbf{B}}_{\text{in}} = -\frac{\mu_0 I}{4}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{\mathbf{k}} \text{ (into the page since } a < b\text{).}$$

Solution 2

There is no contribution from the straight portion of the wire since $d\vec{\mathbf{s}}\times\hat{\mathbf{r}}=0$. For the field of the spiral, we apply Biot-Savart law:

$$d\vec{\mathbf{B}}=\frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2}=\frac{\mu_0 I}{4\pi}\frac{ds\sin\theta}{r^2}\hat{\mathbf{k}}=\frac{\mu_0 I}{4\pi}\frac{(\sqrt{2}\,dr)\sin(3\pi/4)}{r^2}\hat{\mathbf{k}}$$
$$=\frac{\mu_0 I}{4\pi}\frac{(\sqrt{2}\,dr)(1/\sqrt{2})}{r^2}\hat{\mathbf{k}}=\frac{\mu_0 I}{4\pi}\frac{dr}{r^2}\hat{\mathbf{k}}$$

Substituting $r=e^{\theta}$ and $dr=e^{\theta}d\theta$, the above expression becomes

$$d\vec{\mathbf{B}}=\frac{\mu_0 I}{4\pi}\frac{e^{\theta}d\theta}{e^{2\theta}}\hat{\mathbf{k}}=\frac{\mu_0 I}{4\pi}e^{-\theta}d\theta\,\hat{\mathbf{k}}$$

Integrating the angle from *0* to 2π, we obtain

$$\vec{\mathbf{B}}=\frac{\mu_0 I}{4\pi}\hat{\mathbf{k}}\int_0^{2\pi}e^{-\theta}d\theta=\frac{\mu_0 I}{4\pi}\left(1-e^{-2\pi}\right)\hat{\mathbf{k}}$$

Ampere's Law

Solution 1

The enclosed current is given by

$$I_{enc} = \int \vec{J} \cdot d\vec{A} = \int (\alpha r'^2)(2\pi r' dr') = \int 2\pi\alpha r'^3 \, dr'$$

For $r < R$,

$$I_{enc} = \int_0^r 2\pi\alpha \, r'^3 \, dr' = \frac{\pi\alpha \, r^4}{2}$$

Applying Ampere's law, the magnetic field is given by

$$B(2\pi r) = \frac{\mu_0 \pi\alpha r^4}{2}$$

or

$$B = \frac{\mu_0 \alpha}{4} r^3$$

For $r > R$,

$$I_{enc} = \int_0^R 2\pi\alpha r'^3 \, dr' = \frac{\pi\alpha R^4}{2}$$

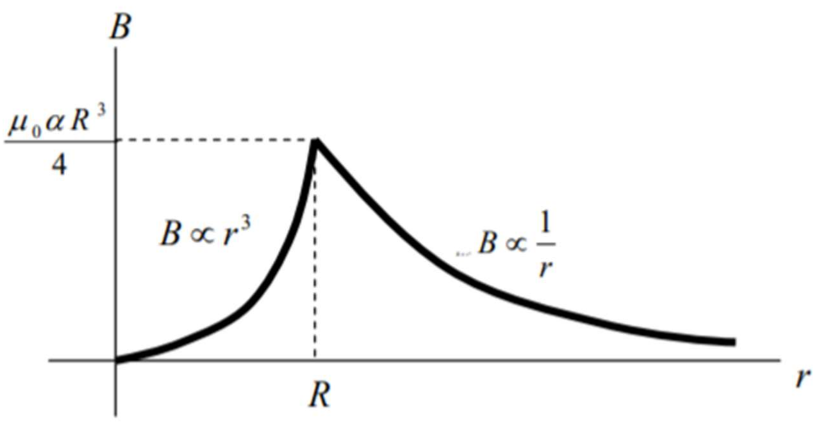
Applying Ampere's law, the magnetic field is given by

$$B(2\pi r) = \frac{\mu_0 \pi\alpha R^4}{2}$$

or

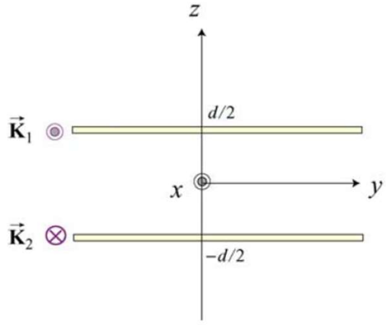
$$B = \frac{\mu_0 \alpha R^4}{4r}$$

(b) Plot the magnitude of the magnetic field as a function of r.

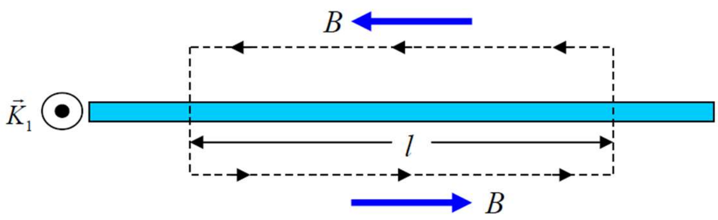


Solution 2

Consider two infinitely large sheets lying in the xy-plane separated by a distance d carrying surface current densities $\vec{\mathbf{K}}_1 = K\hat{\mathbf{i}}$ and $\vec{\mathbf{K}}_2 = -K\hat{\mathbf{i}}$ in the opposite directions, as shown in the figure below (The extent of the sheets in the y direction is infinite.) Note that K is the current per unit width perpendicular to the flow.



(a) Find the magnetic field everywhere due to $\vec{\mathbf{K}}_1$.



Consider the Ampere's loop shown above. The enclosed current is given by

$$I_{enc} = \int \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = Kl$$

Applying Ampere's law, the magnetic field is given by

$$B(2l) = \mu_0 K l \quad \text{or} \quad B = \frac{\mu_0 K}{2}$$

Therefore,

$$\vec{\mathbf{B}}_1 = \begin{cases} -\frac{\mu_0 K}{2}\hat{\mathbf{j}}, & z > \frac{d}{2} \\ \frac{\mu_0 K}{2}\hat{\mathbf{j}}, & z < \frac{d}{2} \end{cases}$$

(b) Find the magnetic field everywhere due to $\vec{\mathbf{K}}_2$.

The result is the same as part (a) except for the direction of the current:

$$\vec{\mathbf{B}}_2 = \begin{cases} \frac{\mu_0 K}{2}\hat{\mathbf{j}}, & z > -\frac{d}{2} \\ -\frac{\mu_0 K}{2}\hat{\mathbf{j}}, & z < -\frac{d}{2} \end{cases}$$

(c) Applying superposition principle, find the magnetic field everywhere due to both current sheets.

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \begin{cases} \mu_0 K\,\hat{\mathbf{j}}, & -\frac{d}{2} < z < \frac{d}{2} \\ 0, & |z| > \frac{d}{2} \end{cases}$$

(d) How would your answer in (c) change if both currents were running in the same direction, with $\vec{\mathbf{K}}_1 = \vec{\mathbf{K}}_2 = K\hat{\mathbf{i}}$?

In this case, $\vec{\mathbf{B}}_1$ remains the same but

$$\vec{\mathbf{B}}_2 = \begin{cases} -\dfrac{\mu_0 K}{2}\hat{\mathbf{j}}, & z > -\dfrac{d}{2} \\ \dfrac{\mu_0 K}{2}\hat{\mathbf{j}}, & z < -\dfrac{d}{2} \end{cases}$$

Therefore,

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \begin{cases} -\mu_0 K\,\hat{\mathbf{j}}, & z > \dfrac{d}{2} \\ 0, & -\dfrac{d}{2} < z < \dfrac{d}{2} \\ \mu_0 K\,\hat{\mathbf{j}}, & z < -\dfrac{d}{2} \end{cases}$$