Lecture 9

Calculate the self inductance of a solenoid with (n - turns, l - length & a - radius)

$$\phi = (\mu_0 n I)\pi a^2 n l$$

$$\therefore L = \mu_0 n^2 \pi a^2 l$$

If we now use this solenoid in a circuit with variable current, the voltage drop $= -L\frac{dI}{dt}$.

It is not necessary that the inductor is a solenoid. It can be any structure with some self inductance.

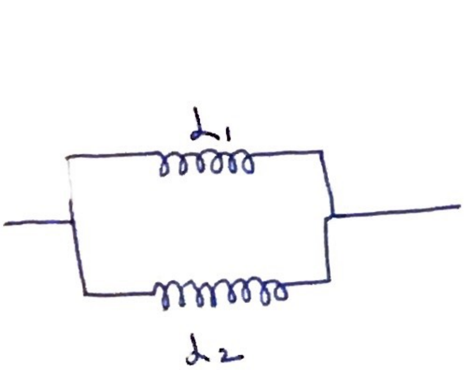
Using the above, let us see how we can combine inductors in circuits.

The voltage drop is the sum of voltage drop across individual inductors.
But current is equal through both.

$$V_{tot} = V_1 + V_2$$

$$\therefore L\frac{dI}{dt} = L_1\frac{dI}{dt} + L_2\frac{dI}{dt}$$

$$\therefore L = L_1 + L_2$$



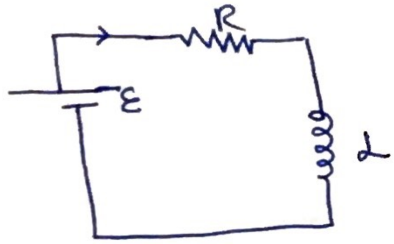
The current in each arm be I_1 & I_2
& voltage is same across both.

$$\therefore I = I_1 + I_2$$

$$\Rightarrow \cancel{L}\frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$

$$\Rightarrow \frac{V}{L} = V/L_1 + V/L_2 \Rightarrow \frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}$$

Let us now study a simple circuit.



At some arbitrary time t.

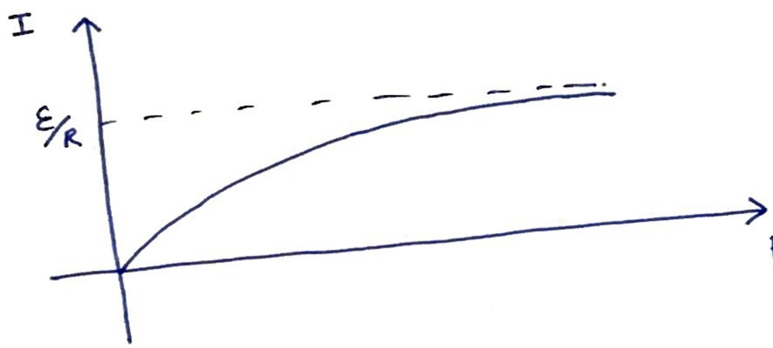
$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

$$\Rightarrow L\,dI/dt = \mathcal{E} - IR$$

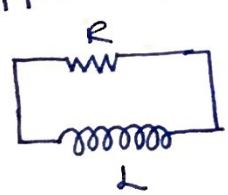
$$\Rightarrow dI/\mathcal{E} - IR = \frac{dt}{L}$$

$$\Rightarrow I = \mathcal{E}/R\,(1 - e^{-R/L\, t})$$

$$\therefore \tau = L/R$$



What happens if I now discharge the ~~circuit?~~ inductor?



$$-L\frac{dI}{dt} - IR = 0$$

$$\Rightarrow -L\,dI/dt = IR$$

$$\Rightarrow I = I_0\, e^{-Rt/L}$$

Using the information of the energy dissipated by the resistor, calculate the energy stored in the inductor.

$$U = \int_0^\infty R I^2\, dt$$

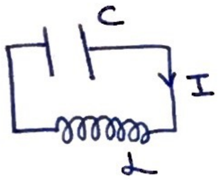
$$= \int_0^\infty R I_0^2\, e^{-2Rt/L} = R I_0^2 \frac{L}{2R} = \frac{1}{2} L I_0^2$$

I wont solve it here as it has already been solved in the class

$$\text{Energy} = \frac{B^2}{2\mu_0}$$

Recall :- Similarly to the energy stored in electric fields, this is the energy required to create the fields in the first place.

Circuit 2



Let us take a capacitor that is initially charged.

$$-L\frac{dI}{dt} + Q/C = 0$$

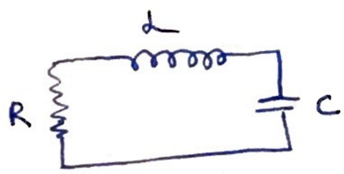
$$\Rightarrow L\frac{dI}{dt} = Q/C$$

$$\Rightarrow d^2Q/dt^2 = \frac{Q}{LC}$$

$$\therefore Q = Q_0 \cos(\omega t + \phi)$$

$$\text{with } \omega = \frac{1}{\sqrt{LC}}$$

Circuit 3



Let us take the capacitor to be initially charged

$$-IR - L\frac{dI}{dt} + Q/C = 0$$

$$\Rightarrow L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + Q/C = 0$$

$$\Rightarrow d^2Q/dt^2 + R/L \, dQ/dt + Q/C = 0$$

This requires a Q of the form

$$Q = Q_0 e^{-Rt/2L} \cos(\omega t + \phi)$$

$$\text{with } \omega = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}}$$

Another helpful form of the same equation is

$$Q = e^{-Rt/2L}\left[A e^{i\omega t} + B e^{-i\omega t}\right]$$

We further define a quantity that helps ~~defin~~ describe the circuits.

Q (quality factor) $= 2\pi \times$ no of cycles it takes for the energy stored to drop by $1/e$

(less the damping → higher the Q)

In our above example, energy stored is in the capacitor initially.

$$\therefore E_c \propto Q^2$$

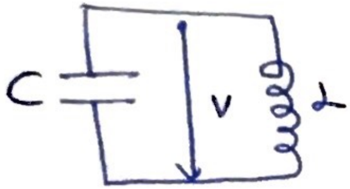
$$\therefore E_c \propto e^{-\frac{\cancel{2}Rt}{\cancel{2}L}}$$

$$\therefore E_c \rightarrow \tfrac{1}{e} E_{c_0} \quad \text{at } t = L/R$$

$$\therefore \text{No of radians} = \omega t = \frac{\omega L}{R}$$

Alternatively Q may also be expressed in cycles and then

$$Q = \frac{\omega L}{2\pi R}$$



I set up the circuit such that initial conditions have voltage V_0 & varies as $V_0 \cos \omega t$. What is the energy stored at $t = 0$ & $t = \pi/2\omega$

For voltage $V_0 \cos \omega t$, the capacitor plate must hold

$$Q = C V_0 \cos(\omega t) \text{ charge.}$$

$\therefore$ Current through inductor $= C V_0 \omega \sin(\omega t)$

$\therefore$ there is no energy in the inductor.

$\therefore$ Energy $= \frac{1}{2} C V_0^2 / 2$

At $t = \pi/2\omega$

$$Q = 0$$

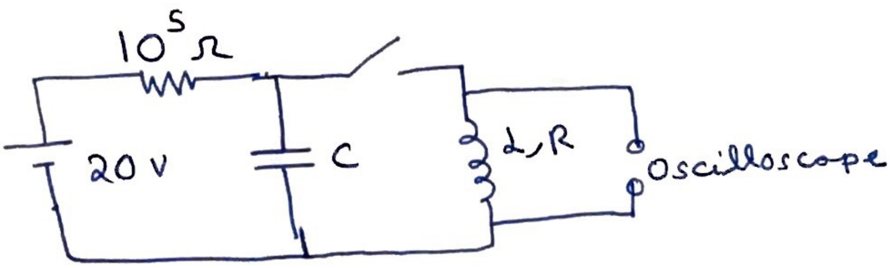
$$I = C V_0 \omega$$

$\therefore$ Energy $= \frac{1}{2} L C^2 V_0^2 \omega^2$

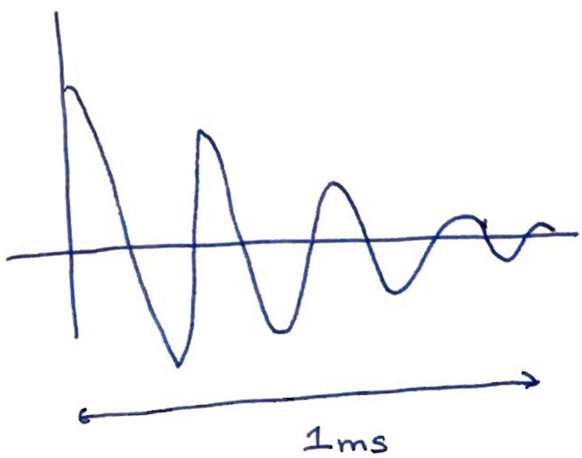
$$= \frac{1}{2} C V_0^2$$

Problem

$L = 0.01$ Henry



Oscilloscope reading



$1/e$ in 0.5 ms

Consider the $10^5\,\Omega$ large enough that no current flows through it.

a) What is the 'C' in the circuit?

b) Estimate 'R'.

c) What is the voltage across the oscilloscope a long time after switch is opened?

a) When the amplitude doesn't become negligible after a few cycles, you can consider $\omega \approx \frac{1}{\sqrt{LC}}$

Using $L = 0.01$ H

& $\omega = 4 \times \frac{2\pi}{T} = 2.5 \times 10^4\ s^{-1}$

$\therefore C = 1.6 \times 10^{-7}$ F

b> As we just studied, voltage goes to $1/e$ in $t = \frac{2L}{R}$

$$\therefore \quad R = \frac{2L}{t}$$

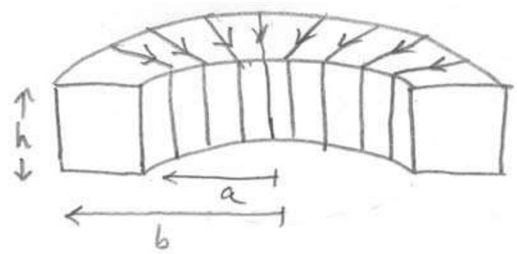
$$= 40\,\Omega$$

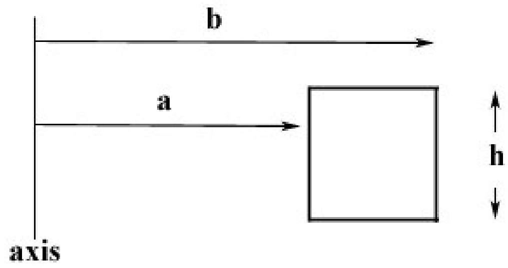
c> After a long time, we essentially have two resistors, $10^5\,\Omega$ & $40\,\Omega$ in series with $20\,V$.

$$\therefore \quad V_{40\Omega} = \left(\frac{40}{10^5 + 40}\right) 20 = 0.008\,V$$

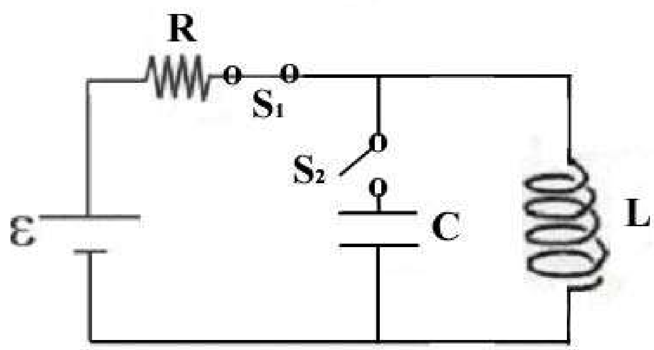
Problem 6:

A toroidoil coil has N turns, and an inner radius a, outer radius b, and height h. The coil has a rectangular cross section shown in the figures below.





The coil is connected via a switch, S_1, to an ideal voltage source with electromotive force $\mathcal{E}$. The circuit has total resistance R. Assume all the self-inductance L in the circuit is due to the coil. At time $t = 0$ S_1 is closed and S_2 remains open.



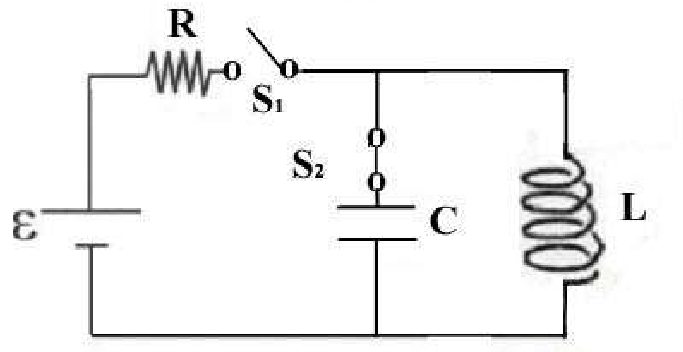
a) When a current I is flowing in the circuit, find an expression for the magnitude of the magnetic field inside the coil as a function of distance r from the axis of the coil.

b) What is the self-inductance L of the coil?

c) What is the current in the circuit a very long time $(t >> L/R)$ after S_1 is closed?

d) How much energy is stored in the magnetic field of the coil a very long time $(t >> L/R)$ after S_1 is closed?

For the next two parts, assume that a very long time $(t >> L/R)$ after the switch S_1 was closed, the voltage source is disconnected from the circuit by opening S_1, and by simultaneously closing S_2 the toroid is connected to a capacitor of capacitance C. Assume there is negligible resistance in this new circuit.

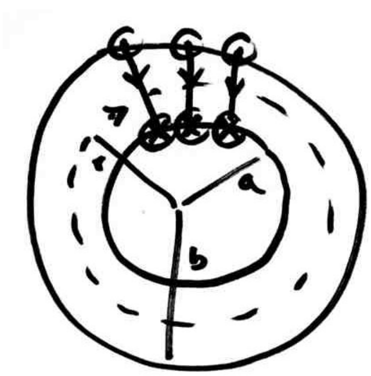


e) What is the maximum amount of charge that will appear on the capacitor?

f) How long will it take for the capacitor to first reach a maximal charge after S_2 has been closed?

Problem 6 Solution:

(a) The magnetic field is zero for $r < a$ and $r > b$. Choose a circle of radius r with $a < r < b$ for your Amperian loop (see figure below).



Then the left hand side of Ampere's Law, $\oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{r}} = \mu_0 I_{enc}$, becomes $\oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{r}} = B2\pi r$.

Since all N turns cuts through the Amperian circle, the right hand side of Ampere's law becomes $\mu_0 I_{enc} = \mu_0 NI$. So setting the two sides equal yields

$$B 2\pi r = \mu_0 N I \,.$$

Thus the magnitude of the magnetic field in the toroid is non-uniform (varies with distance r from the center) and is equal to is

$$B = \begin{cases} 0, & r < a \text{ and } r > b \\ \dfrac{\mu_0 N\, I}{2\pi r}, & a < r < b \end{cases}$$

The field points in the clockwise direction when viewed from above

(b) We first need to find the magnitude of the magnetic flux through the toroid. We need to integrate the non-uniform magnetic field over the cross sectional area of one turn so we use for the area element $da = hdr$. Then the

$$\left| \int_{toroid} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \right| = \left| N \int_{one\ turn} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \right| = \left| N \int_{r=a}^{r=b} \frac{\mu_0 N I}{2\pi r} h dr \right| = \frac{\mu_0 N^2 h \ln(b/a)}{2\pi} I \,.$$

The magnetic flux through the toroid is proportional to the current,

$$N \int_{one turn} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = LI \,.$$

The constant of proportionality is called the self-inductance,

$$L = \left| N \int_{one turn} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \Big/ I \right| .$$

The unit of self-inductance is the henry, [H], $[H] = [T \cdot m^2] / [A]$ and is given by

$$L_{toroid} = \frac{\mu_0 N^2 h \ln(b/a)}{2\pi}$$

(c) Solution: A very long time after the switch S_1 was closed, the current is steady so the inductor acts like a short and the current in the circuit is

$$I = \varepsilon / R$$

(d) The energy stored in the magnetic field is equal to

$$U_{mag} = \frac{1}{2}LI^2 = \frac{\mu_0 N^2 h \ln(b/a)}{4\pi R^2}\varepsilon^2 .$$

(e) When the switch S_2 is closed the current in the circuit is $I = \varepsilon / R.$ The maximum amount of charge occurs when all the magnetic energy is converted to electrical energy

$$U_{elec} = \frac{Q_{\max}{}^2}{2C} = U_{mag,0} = \frac{1}{2}LI^2 = \frac{\mu_0 N^2 h \ln(b/a)}{4\pi R^2}\varepsilon^2 .$$

We can solve the above equation for the maximal charge on the capacitor

$$Q_{\max} = \sqrt{2CU_{mag,0}} = \sqrt{CLI_0{}^2} = \sqrt{\frac{2C\mu_0 N^2 h \ln(b/a)}{4\pi}}\,\varepsilon / R.$$

(f) It will take one quarter cycle, or

$$t = \frac{1}{4}T = \frac{1}{4}\frac{2\pi}{\omega_0} = \frac{\pi}{2}\sqrt{LC} = \frac{\pi}{2}\sqrt{\frac{C\mu_0 N^2 h \ln(b/a)}{2\pi}}$$