Lecture 7

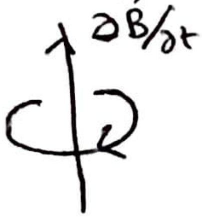
So far,

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad \vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$$

By comparing,

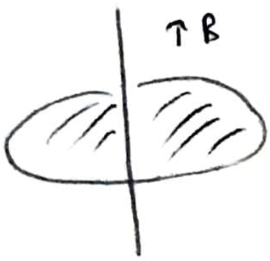
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ $\Rightarrow$  $\vec{B}$ in $\hat{\theta}$ direction

$\therefore$ $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$ $\Rightarrow$ 

$\vec{E}$ in $-\hat{\theta}$ direction.

Problem 1

Consider a disc of radius - a in a region of magnetic field $\vec{B}$ acting upwards. The disc consists of stationary charges with surface charge density σ. The magnetic field is slowly turned off. Find the final angular momentum.



We know $\vec{E}$ is in θ direction from our discussion above and using radial symmetry.

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\phi}{dt}$$

$$\Rightarrow E_\theta\, 2\pi r = -\pi r^2 \frac{dB}{dt}$$

$$\Rightarrow E_\theta = -\frac{\pi r}{2}\frac{dB}{dt}$$

Torque for the ring $= \vec{r} \times \vec{F}$

$$= r\left(\sigma 2\pi r\, dr \left(-\frac{\pi r}{2}\right)\frac{dB}{dt}\right)$$

∴ Total Momentum for ring $= \int \tau_{(r)}\, dt$

$$= -\pi^2 r^3 dr\, \sigma\, (0 - B)$$

$$= \pi^2 r^3 dr\, \sigma B$$

∴ L over whole disc $= \dfrac{\sigma B \pi^2 a^4}{4}$

Where does this angular momentum come from?

energy ?

Problem 2

A long solenoid carries a current $I(t)$. Find $\vec{E}$ inside and outside in quasi-static approximation.

$$\vec{B} = \mu_0 n I(t)\, \hat{z}, \quad r < a$$

$$= 0, \quad r > a$$

∴ Inside for faraday loop (?) of $r < a$

$$\phi = B\pi r^2$$

$$\Rightarrow \oint \vec{E} \cdot d\vec{l} = -\frac{d\phi}{dt}$$

$$\Rightarrow |\vec{E}|\, 2\pi r = -\mu_0 n \pi r^2 \frac{dI}{dt}$$

$$\Rightarrow \vec{E} = -\frac{\mu_0 n r}{2}\frac{dI}{dt}\,\hat{\theta}$$

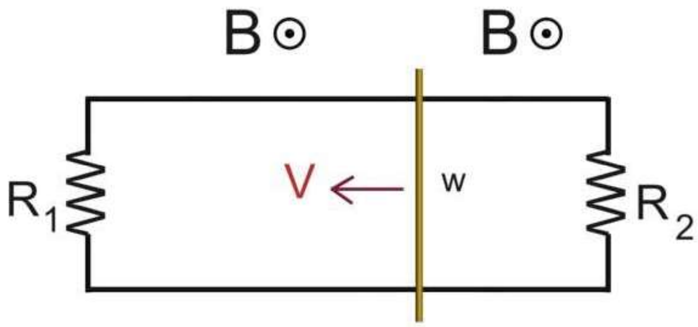
Outside : for faraday loop of $r > a$

$$\phi = B\pi a^2$$

$$\Rightarrow E = -\frac{\mu_0 n \pi a^2}{2\pi r}\frac{dI}{dt}\,\hat{\theta}$$

Problem 5:

A conducting rod with zero resistance and length w slides without friction on two parallel perfectly conducting wires. Resistors R_1 and R_2 are connected across the ends of the wires to form a circuit, as shown. A constant magnetic field **B** is directed out of the page. In computing magnetic flux through any surface, take the surface normal to be out of the page, parallel to **B**.



(a) The magnetic flux in the right loop of the circuit shown is (circle one)
  1) decreasing
  2) increasing.

What is the magnitude of the rate of change of the magnetic flux through the right loop?

(b) What is the current flowing through the resistor R_2 in the right hand loop of the circuit shown? Give its magnitude and indicate its direction on the figure.

(c) The magnetic flux in the left loop of the circuit shown is (circle one)
  1) decreasing
  2) increasing.

What is the magnitude of the rate of change of the magnetic flux through the right loop?

(d) What is the current flowing through the resistor R_1 in the left hand loop of the circuit shown? Gives its magnitude and indicate its direction on the figure.

(e) What is the magnitude and direction of the magnetic force exerted on this rod?

Problem 5 Solutions:

(a) The magnetic flux in the right loop of the circuit shown is (circle one)

  2) increasing.

What is the magnitude of the rate of change of the magnetic flux through the right loop?

$$\frac{d\Phi(t)}{dt} = \frac{d}{dt}BA = B\frac{d}{dt}A = BwV$$

(b) The flux out of the page is increasing so the current is clockwise to make a flux into the page. The magnitude we can get from Faraday:

$$I = \frac{|\varepsilon|}{R_2} = \frac{1}{R_2}\frac{d\Phi(t)}{dt} = \frac{BwV}{R_2}$$

(c) The magnetic flux in the left loop of the circuit shown is (circle one)

1) decreasing

What is the magnitude of the rate of change of the magnetic flux through the right loop?

$$\frac{d\Phi(t)}{dt} = \frac{d}{dt}BA = B\frac{d}{dt}A = -BwV$$

"Magnitude" is ambiguous – either a positive or negative number will do here. I use the negative sign to indicate that the flux is decreasing.

(d) The flux out of the page is decreasing so the current is counterclockwise to make a flux out of the page to make up for the loss. The magnitude we can get from Faraday:

$$I = \frac{|\varepsilon|}{R_1} = \frac{1}{R_1}\frac{d\Phi(t)}{dt} = \frac{BwV}{R_1}$$

(e) The total current through the rod is the sum of the two currents (they both go up through the rod). Using the right hand rule on $\vec{\mathbf{F}} = I\vec{\mathbf{L}} \times \vec{\mathbf{B}}$ we see the force is to the **right**. You could also get this directly from Lenz. The magnitude of the force is:

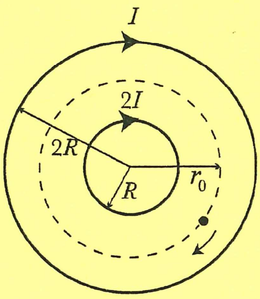
$$F = \left|I\vec{\mathbf{L}} \times \vec{\mathbf{B}}\right| = ILB = \left(BwV\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right)wB = \boxed{B^2w^2V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

Problem 2: Two Solenoids (6 points)

An infinitely long solenoid contains another coaxial solenoid, also infinitely long. The radius R of the inner solenoid is half the radius of the outer solenoid. Both solenoids have the same number n of coil turns per unit length. The current is switched on for both solenoids at $t = 0$. The current then increases linearly with time:

$$I(t) = kt, \text{ for } t > 0.$$

The inner solenoid's current is twice as large as the current of the outer one, and their directions are the same.



(a) (1 point) Find the magnitude and direction of the magnetic field $\vec{B}$ as a function of r (the distance from the axis of the solenoids). Express your answer in terms of n, k, and and t.

(b) (1 point) Find the magnetic flux Φ through a surface whose boundary is a circle of radius r_0. Let $R < r_0 < 2R$ (see the dashed curve in the illustration).

(c) (2 points) Find the EMF $\mathcal{E}$ that would act on a particle moving along the circular trajectory of radius r_0. Give the magnitude and direction of the induced electric field $\vec{E}(r_0)$.

A particle of charge q and mass m initially moves along a circular orbit of radius r_0, concentric with the axis of the solenoids. Although the EMF causes the particle to accelerate tangentially, it is possible for the particle to stay on a circular orbit, provided that the centripetal force increases accordingly.

For the following questions, assume that the particle moves with non-relativistic speed $v \ll c$ throughout.

(d) (1 point) Write down an equation for $B(r_0)$ in terms of the tangential velocity v of the charged particle, in the case in which the particle keeps moving on a circular orbit of radius r_0.

(e) (1 point) If the particle starts at rest, find the value of r_0 for which it will remain on a circular orbit.
Hint: Your final answer should depend only on R.

That is, the driving voltage starts at V_0 and grows with time, while $|\mathcal{E}_{back}|$ starts at $\frac{M_{21}^2 V_0}{R_1 L\tau}$ and decreases with time. Therefore, the sufficient condition for being able to neglect the back EMF altogether is:

$$\frac{M_{21}^2 V_0}{R_1 L\tau} \ll V_0 \ ,$$

or, equivalently,

$$R_1 L\tau \gg M_{21}^2 \ .$$

Problem 2: Two Solenoids (6 points)

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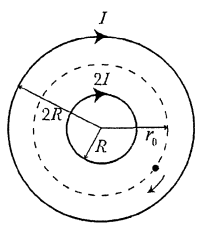
$$I(t) = kt, \text{ for } t > 0.$$

The inner solenoid's current is twice as large as the current of the outer one, and their directions are the same.



(a) (1 point) Find the magnitude and direction of the magnetic field $\vec{B}$ as a function of r (the distance from the axis of the solenoids). Express your answer in terms of n, k, and and t.

Solution: For an infinite solenoid of current I and n turns per unit length, the field outside of the coil vanishes, while the field inside is constant, with magnitude $4\pi In/c$. Before the current is turned on the magnetic field is obviously zero everywhere. By superposition, the magnitude of the magnetic field for $t > 0$ is:

$$B(r) = \begin{cases} 12\pi kt/c & \text{for } r < R \\ 4\pi kt/c & \text{for } R < r < 2R \\ 0 & \text{for } r > 2R \end{cases} .$$

By the right-hand rule, for the arrangement shown in the problem's illustration, the magnetic field must point into the plane of the page.

(b) (1 point) Find the magnetic flux Φ through a surface whose boundary is a circle of radius r_0. Let $R < r_0 < 2R$ (see the dashed curve in the illustration).

Solution: The total flux is simply

$$\Phi = \int \vec{B} \cdot d\vec{a} = \frac{4\pi kt}{c}\left(2\pi R^2 + \pi r_0^2\right) = \underline{\frac{4\pi^2 kt}{c}\left(2R^2 + r_0^2\right)} \ .$$

(c) (2 points) Find the EMF $\mathcal{E}$ that would act on a particle moving along the circular trajectory of radius r_0. Give the magnitude and direction of the induced electric field $\vec{E}(r_0)$.

Solution: By Faraday's law of induction:

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{s} = -\frac{1}{c}\frac{d\Phi}{dt} = \underline{-\frac{4\pi^2 k}{c^2}\left(2R^2 + r_0^2\right)} \ .$$

The sign of $\mathcal{E}$ depends on our choice of orientation for the contour of integration. In this case we have taken the contour to run clockwise in the illustration, which gives a negative EMF.

Since there is no net electrical charge anywhere, Gauss's Law tells us that $\vec{\nabla} \cdot \vec{E} = 0$. Therefore we know that the electric field is purely circumferential on the plane of the page in the illustration.[1]

Thus

$$\oint \vec{E} \cdot d\vec{s} = E(r_0) \cdot 2\pi r_0 \ ,$$

which, using out result for $\mathcal{E}$, implies that

$$\underline{E(r_0) = \frac{2\pi k}{c^2 r_0}\left(2R^2 + r_0^2\right)} \ .$$

The induced electric field points counterclockwise in the illustration. This agrees with Lenz's Law: The magnetic flux Φ is increasing into the plane of the page. Therefore, the EMF would create a current which would produce a magnetic flux *out* of the plane of the page. By the right-hand rule, this must be a counterclockwise current.

A particle of charge q and mass m initially moves along a circular orbit of radius r_0, concentric with the axis of the solenoids. Although the EMF causes the particle to accelerate tangentially, it is possible for the particle to stay on a circular orbit, provided that the centripetal force increases accordingly.

For the following questions, assume that the particle moves with non-relativistic speed $v \ll c$ throughout.

[1] Strictly, we also need to show that the field has no component perpendicular to the plane of the page. The cylindrical symmetry of the problem and the usual physical requirement that the electrical field die off at infinity tell us that this is the case.

(d) **(1 point)** Write down an equation for $B(r_0)$ in terms of the tangential velocity v of the charged particle, in the case in which the particle keeps moving on a circular orbit of radius r_0.

Solution: As in the case of cyclotron motion, we simply set the magnetic force on the charge particle equal to the centripetal force required to keep the particle in a circular orbit of radius r_0. Non-relativistically:

$$q\frac{v}{c}B(r_0) = \frac{mv^2}{r_0} \ ,$$

which implies that

$$\underline{B(r_0)= \frac{mvc}{qr_0}} \ .$$

(e) **(1 point)** If the particle starts at rest, find the value of r_0 for which it will remain on a circular orbit.
Hint: Your final answer should depend only on R.

Solution: As the EMF accelerates the charge, we have, non-relativistically, that:

$$v = \frac{qE(r_0)t}{m} = \frac{2\pi kqt}{mc^2 r_0}\left(2R^2 + r_0^2\right) \ .$$

Using the results of parts **(a)** and **(d)** we obtain that:

$$\begin{aligned} \frac{mvc}{qr_0} &= B(r_0) \\ \frac{2\pi kt}{r_0^2 c}\left(2R^2 + r_0^2\right) &= \frac{4\pi kt}{c} \\ r_0 &= \underline{\sqrt{2}R} \ . \end{aligned}$$