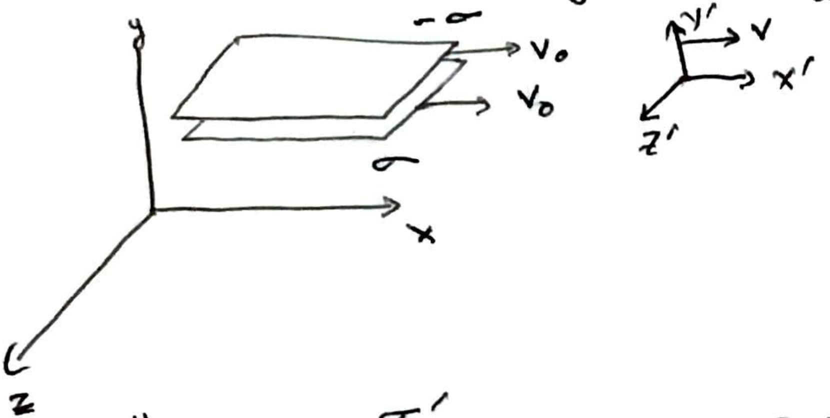
Lecture 5

Lets quickly review what Dave covered in class. We will not cover the Hall effect but instead focus on problems.

We saw last rec session already ~~what~~ how $E_{\parallel}$ & $E_{\perp}$ transform under a lorentz boost.

To finish our analysis for the same setup with $\vec{B}$ fields. and relate them to ~~B~~ $\vec{E}$ fields.

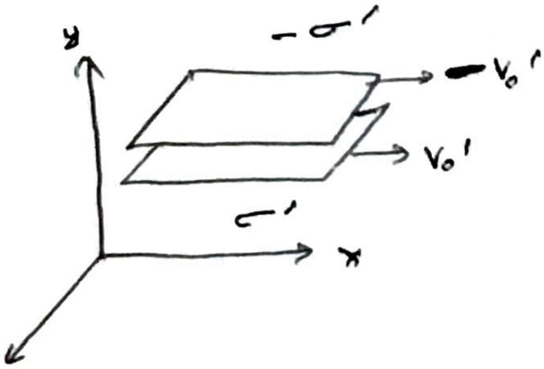


$$E_y = \frac{\sigma}{\varepsilon_0}\hat{y}$$

$$\& \quad K_1 = \sigma v_0 \hat{x} \text{ (for bottom sheet)}$$

$$K_2 = \sigma v_0 (-\hat{x}) \text{ (for top sheet)}$$

$$B_z = \mu_0 K \hat{z}$$



$$v_0' = (v_0 - v) / (1 - v_0 v/c^2)$$

$$\gamma_0' = \gamma_0 \gamma (1 - \beta_0 \beta)$$

$$\sigma' = \sigma_{rest}\gamma_0' = (\sigma/\gamma_0)\gamma_0'$$

$$\therefore E_y' = \sigma'/\varepsilon_0 = \sigma/\varepsilon_0 (\gamma_0'/\gamma_0)$$

$$= \frac{\sigma}{\varepsilon_0}\gamma(1 - \beta_0\beta)$$

$$= \gamma\left(\frac{\sigma}{\varepsilon_0} - v\frac{\sigma \mu_0 v_0}{\mu_0 \varepsilon_0 c^2}\right)$$

using $\mu_0 \varepsilon_0 = 1/c^2$

$$= \gamma(E_y - vB_z)$$

Similarly,

~~B~~

$$B_z' = \mu_0 K' = \mu_0 \sigma' v_0'$$

$$= \mu_0 \sigma \gamma \cancel{(1-\beta_0\beta)}\, c \frac{(\beta_0 - \beta)}{\cancel{(1-\beta_0\beta)}}$$

$$= \gamma \left(\frac{v_0}{c} \mu_0 \sigma c - \frac{v}{c} \mu_0 \sigma c \right)$$

$$= \gamma \left(\underbrace{v_0 \mu_0 \sigma}_{B_z} - v \mu_0 \epsilon_0 \underbrace{\frac{\sigma}{\epsilon_0}}_{\to E_y} \right)$$

$$= \gamma \left(B_z - \frac{v}{c^2} E_y \right)$$

Using the same method for E_z & B_y, we get

$$E_z' = \gamma (E_z + v B_y) \quad \& \quad B_y' = \gamma \left(B_y + \left(\frac{v}{c^2}\right) E_z \right)$$

∴ Summarizing,

$$E_x' = E_x, \quad E_y' = \gamma(E_y - vB_z), \quad E_z' = \gamma(E_z + vB_y)$$

$$B_x' = B_x, \quad B_y' = \gamma(B_y + v/c^2\, E_z), \quad B_z' = \gamma(B_z - v/c^2\, E_y)$$

or

$$E_{\parallel}' = E_{\parallel}, \quad \vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp})$$

$$B_{\parallel}' = B_{\parallel}, \quad \vec{B}_{\perp}' = \gamma\left(\vec{B}_{\perp} - (\vec{v}/c^2) \times \vec{E}_{\perp}\right)$$

While Gaussian units are more natural for EM, we will continue with SI.

Special Cases

→ In frame where $B_{\parallel} = 0$ & $B_{\perp} = 0$

$$B_{\parallel}' = 0 \quad \& \quad E_{\perp}' = \gamma E_{\perp}$$

$$B_{\perp}' = \gamma\left(-\vec{v}/c^2 \times \vec{E}_{\perp}\right) = -\vec{v}/c^2 \times \vec{E}_{\perp}'$$

as $B_{\parallel}' = 0$ & $\vec{v} \times \vec{E}_{\parallel} = 0$

$$B' = -(\vec{v}/c^2) \times \vec{E}'$$

→ In frame where $E_\perp = 0$ & $E_{\parallel} = 0$

$$E'_\perp = \gamma(\vec{v} \times \vec{B}_\perp), \quad \vec{E}'_{\parallel} = 0$$

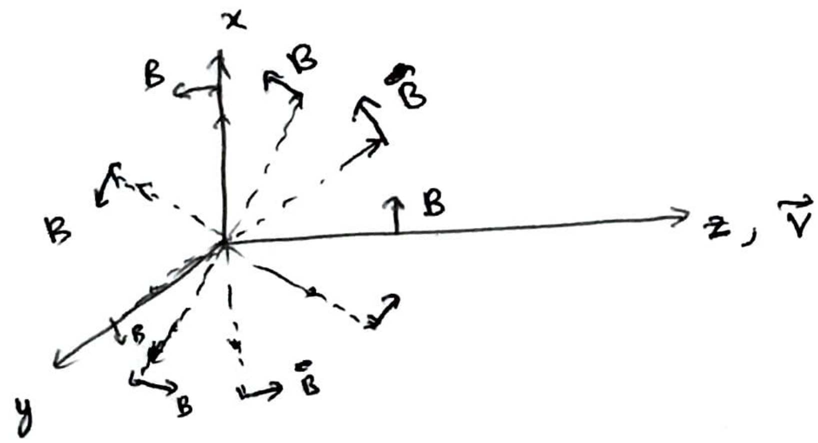
$$B'_{\parallel} = B_{\parallel}, \quad B'_\perp = \gamma B_\perp$$

$$\vec{E}' = \vec{v} \times \vec{B}'$$

Using the above results, for a charged particle moving at near light speeds, the $\vec{E}'$ is concentrad an a disc $\perp$ to the direction of motion

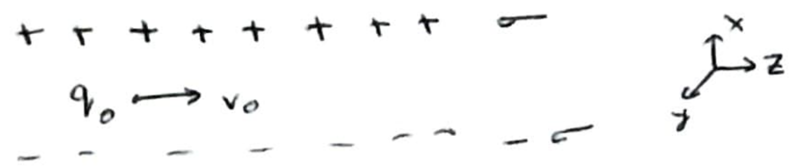
$\therefore$ Using $\vec{B}' = (-\vec{v}/c^2) \times \vec{E}$

$\vec{B}'$ is $\perp$ to both $\vec{E}'$ & $\vec{v}$



(I also suggest that you study Rowland's experiment from Purcell 6.8 to see the immense efforts taken by scientists to study the mysteries of EM)

Problem 1 Relativistic Capacitor



a) What is the $\vec{E}$ & $\vec{B}$ by the capacitor in the ground frame?

b) What is the $\vec{E}$ & $\vec{B}$ in the frame of the charge?

Use Lorentz Transformation and first principle approaches to verify your answer.

a) $\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{x}$

$\vec{B} = 0$

b) $E'_{\parallel} = 0$

$E'_{\perp} = \gamma E_{\perp}(-\hat{x}) = -\frac{\gamma\sigma}{\varepsilon_0}\hat{x}$

$B'_{\parallel} = 0$

$\vec{B}'_{\perp} = -\frac{\vec{v}}{c^2} \times \vec{E}'_{\perp} = \frac{v_0}{c^2}\gamma\frac{\sigma}{\varepsilon_0}\hat{y}$

From first principles

$\sigma' = \gamma\sigma$

$E' = \gamma\frac{\sigma}{\varepsilon_0}(-\hat{x})$

$K' = -\sigma' v_0 \hat{z} = -\gamma\sigma v_0 \hat{z}$

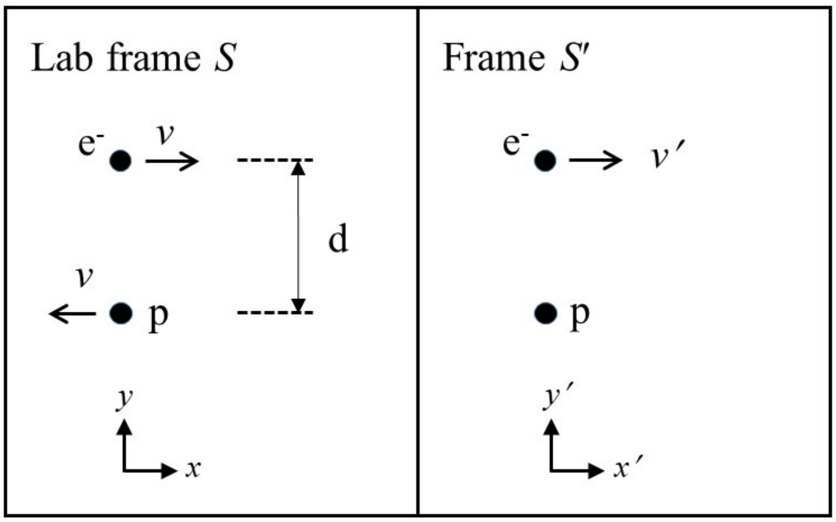
$\vec{B}' = \mu_0 K' \hat{y}$

$= \mu_0 \gamma \sigma v_0 \hat{y}$

$= \frac{v_0}{c^2}\gamma\frac{\sigma}{\varepsilon_0}\hat{y}$

Problem 1 – Forces between relativistic particles

As seen in the lab frame *S*, a proton and electron travel in opposite directions parallel to the *x*-axis. Each has a constant relativistic (i.e. not small relative to the speed of light *c*) speed *v* and they are separated by a perpendicular distance *d* in the *y*-direction. Define the primed frame *S*′ to be the frame in which the proton is at rest and the electron moves at a speed *v*′. Give your answers to this problem in terms of *e*, *v*, and *d*.



(a) Compute the electric forces $\vec{F}_e$ and $\vec{F}_p$ felt by the electron and proton respectively in the lab frame *S* at the instant when both particles are on the *y*-axis. [2pts]

(b) Find the electric forces $\vec{F}_e'$ and $\vec{F}_p'$ felt by the electron and proton respectively in *S*′ at the instant when both particles are on the *y*′-axis. [2pts]

(c) Transform the force $\vec{F}_p'$ [from part (b)] into frame *S*. Compute the difference with $\vec{F}_p$ [from part (a)] and obtain the magnetic force $\vec{F}_m$ felt by the proton in *S* at this instant. [1pt]

Question 1

a) Recall the formula for the electric field of a moving charge:

$$\mathbf{E} = \frac{Q}{r^2} \frac{1-\beta^2}{\left(1-\beta^2 \sin^2\theta\right)^{3/2}} \hat{\mathbf{r}} \tag{1}$$

Using this with ($\theta = 90°$), and F = qE, we get:

$$\mathbf{F}_e = \frac{-e^2}{d^2} \frac{1}{\sqrt{1-\beta^2}} \hat{\mathbf{y}} = -\frac{e^2\gamma}{d^2} \hat{\mathbf{y}} \tag{2}$$

$$\mathbf{F}_p = \frac{-e^2}{d^2} \frac{1}{\sqrt{1-\beta^2}} (-\hat{\mathbf{y}}) = \frac{e^2\gamma}{d^2} \hat{\mathbf{y}} \tag{3}$$

Not surprisingly, the electric forces are attractive in direction.

b) The field of the proton is simple in S', just that of a point charge. Furthermore, since y is a perpendicular direction, it is unchanged by the Lorentz transformation. Thus $d = d'$ and $\hat{y}' = \hat{y}$. Hence:

$$\mathbf{F}'_e = (-e) \frac{e}{(d')^2} \hat{\mathbf{y}}' = -\frac{e^2}{d^2} \hat{\mathbf{y}} \tag{4}$$

The field of the electron is given by the equation for a moving charge, where addition of velocities gives:

$$\beta' = \frac{\beta+\beta}{1+\beta\beta} = \frac{2\beta}{1+\beta^2} \tag{5}$$

$$\gamma' = \frac{1}{\sqrt{1-\beta'^2}} = \frac{1}{\sqrt{1-\left(\frac{2\beta}{1+\beta^2}\right)^2}} \tag{6}$$

$$\gamma' = \frac{1}{\sqrt{\frac{(1+\beta^2)^2-4\beta^2}{(1+\beta^2)^2}}} = \frac{1+\beta^2}{\sqrt{1-2\beta^2+\beta^4}} \tag{7}$$

$$\gamma' = \frac{1+\beta^2}{1-\beta^2} = \gamma^2\left(1+\beta^2\right) \tag{8}$$

Thus using the field of the moving charge, we find the force on the proton:

$$\mathbf{F}'_p = \frac{-e^2}{(d')^2} \frac{1}{\sqrt{1-\beta'^2}} \left(-\hat{\mathbf{y}}'\right) \tag{9}$$

$$\mathbf{F}'_p = \frac{e^2\gamma'}{d^2} \hat{\mathbf{y}} = \frac{e^2\gamma^2\left(1+\beta^2\right)}{d^2} \hat{\mathbf{y}} \tag{10}$$

c) In the frame S, we can express the total force on the proton as a sum of electric and magnetic components:

$$\mathbf{F}_T = \mathbf{F}_{\text{ elec }} + \mathbf{F}_{\text{ mag}} \tag{11}$$

$$\mathbf{F}_m = \mathbf{F}_T - \mathbf{F}_p \tag{12}$$

Since the $\mathbf{F}'_T$ is a perpendicular force, we use $F_\perp = \frac{1}{\gamma} F'_\perp$ to get:

$$\mathbf{F}_T = \frac{\gamma' e^2}{\gamma d^2} \hat{\mathbf{y}} \tag{13}$$

Putting these together gives:

$$\mathrm{F}_m = \mathrm{F}_T - \mathrm{F}_p \tag{14}$$
$$\mathrm{F}_m = \frac{\gamma' e^2}{\gamma d^2} \hat{\mathrm{y}} - \frac{\gamma e^2}{d^2} \hat{\mathrm{y}} \tag{15}$$
$$\mathrm{F}_m = \frac{e^2}{d^2} \hat{\mathrm{y}} \left(\frac{\gamma'}{\gamma} - \gamma \right) \tag{16}$$
$$\mathrm{F}_m = \frac{e^2}{d^2} \hat{\mathrm{y}} \left(\gamma \left(1 + \beta^2 \right) - \gamma \right) \tag{17}$$
$$\mathrm{F}_m = \frac{\gamma \beta^2 e^2}{d^2} \hat{\mathrm{y}} \tag{18}$$

The fact that the magnetic force is attractive can be understood by thinking about the electron and proton as being wires, both of which carry current in the same direction ($-\hat{\mathbf{x}}$). Wires with like current directions are attracted.

An alternative method to find F_m considers the frame S'', the rest frame of the electron. This frame moves with velocity $\beta \hat{x}$. In this frame, it is easy to find E'' at the proton, and the electron generates no magnetic field so $\mathrm{B}'' = 0$ at the proton:

$$\mathbf{E}'' = \frac{e}{d^2} \hat{\mathbf{y}} \tag{19}$$

Using the inverse field transformation equations, we can find B in S:

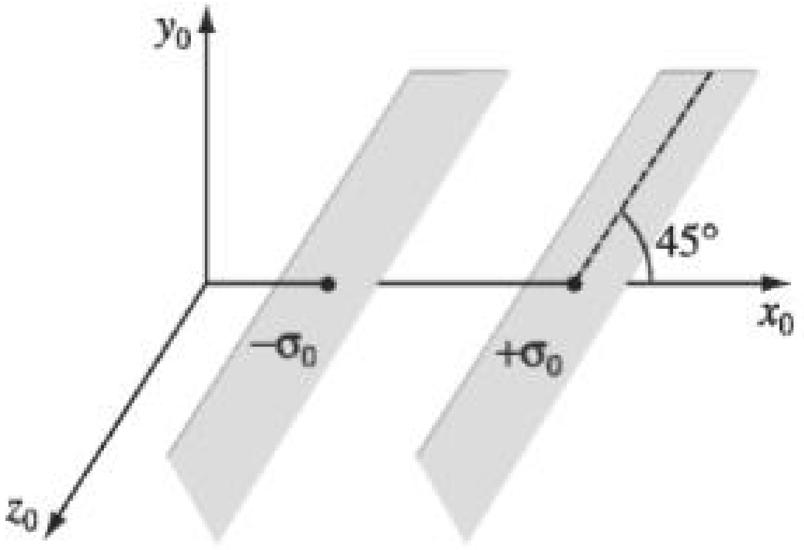
$$\mathrm{B}_\parallel = \mathrm{B}''_\parallel = 0 \tag{20}$$
$$\mathrm{B}_\perp = \gamma \left(\mathrm{B}''_\perp + \beta \times \mathrm{E}''_\perp \right) \tag{21}$$
$$\mathrm{B} = \mathrm{B}_\perp = \gamma (\beta \hat{\mathrm{x}}) \times \left(\frac{e}{d^2} \hat{\mathrm{y}} \right) \tag{22}$$
$$\mathrm{B} = \frac{\gamma \beta e}{d^2} \hat{\mathrm{z}} \tag{23}$$

Next, we use the Lorentz force law to find the magnetic force on the proton:

$$\mathrm{F}_m = q \beta \times \mathrm{B} \tag{24}$$
$$\mathrm{F}_m = e(-\beta \hat{\mathrm{x}}) \times \left(\frac{\gamma \beta e}{d^2} \hat{\mathrm{z}} \right) \tag{25}$$
$$\mathrm{F}_m = \frac{\gamma \beta^2 e^2}{d^2} \hat{\mathrm{y}} \tag{26}$$

Thus, either methods give the same answer.

A parallel-plate capacitor, at rest in $\mathcal{S}_0$ and tilted at a 45° angle to the x_0 axis, carries charge densities $\pm\sigma_0$ on the two plates (Fig. 12.41). System $\mathcal{S}$ is moving to the right at speed v relative to $\mathcal{S}_0$.



(a) Find $\mathbf{E}_0$, the field in $\mathcal{S}_0$.

(b) Find $\mathbf{E}$, the field in $\mathcal{S}$.

(c) What angle do the plates make with the x axis?

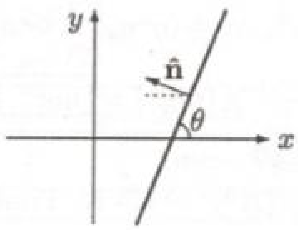
(d) Is the field perpendicular to the plates in $\mathcal{S}$?

(a) Field is σ_0/ϵ_0, and it points perpendicular to the positive plate, so:

$$\mathbf{E}_0 = \frac{\sigma_0}{\epsilon_0}(\cos 45^\circ\,\hat{\mathbf{x}} + \sin 45^\circ\,\hat{\mathbf{y}}) = \boxed{\frac{\sigma_0}{\sqrt{2}\,\epsilon_0}(-\hat{\mathbf{x}} + \hat{\mathbf{y}}).}$$

(b) From Eq. 12.108, $E_x = E_{x_0} = -\frac{\sigma_0}{\sqrt{2}\,\epsilon_0}$; $E_y = \gamma E_{y_0} = \gamma\frac{\sigma_0}{\sqrt{2}\,E_0}$. So $\boxed{\mathbf{E} = \frac{\sigma_0}{\sqrt{2}\,\epsilon_0}(-\hat{\mathbf{x}} + \gamma\,\hat{\mathbf{y}}).}$

(c) From Prob. 12.10: $\tan\theta = \gamma$, so $\boxed{\theta = \tan^{-1}\gamma.}$



(d) Let $\hat{\mathbf{n}}$ be a unit vector perpendicular to the plates in $\mathcal{S}$—evidently $\hat{\mathbf{n}} = -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}$; $|E| = \frac{\sigma_0}{\sqrt{2}\,\epsilon_0}\sqrt{1+\gamma^2}$.

So the angle ϕ between $\hat{\mathbf{n}}$ and $\mathbf{E}$ is:

$$\frac{\mathbf{E}\cdot\hat{\mathbf{n}}}{|E|} = \cos\phi = \frac{1}{\sqrt{1+\gamma^2}}(\sin\theta + \gamma\cos\theta) = \frac{\cos\theta}{\sqrt{1+\gamma^2}}(\tan\theta + \gamma) = \frac{2\gamma}{\sqrt{1+\gamma^2}}\cos\theta$$

But $\gamma = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \sqrt{\frac{1}{\cos^2\theta} - 1} \Rightarrow \frac{1}{\cos}{}^2\theta = \gamma^2 + 1 \Rightarrow \cos\theta = \frac{1}{\sqrt{1+\gamma^2}}$. So $\boxed{\cos\phi = \left(\frac{2\gamma}{1+\gamma^2}\right).}$

Evidently the field is $\boxed{\textit{not}}$ perpendicular to the plates in $\mathcal{S}$.