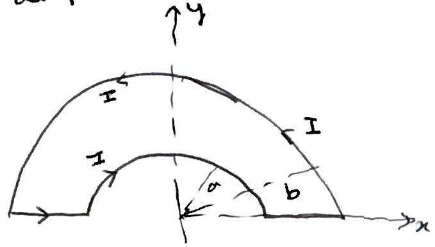
Lecture 3

Problem 1

Find $\vec{B}$ at P



Answer

There is no magnetic field due to the straight segments because point B is along the lines.

$$\vec{B}_{out} = \frac{\mu_0}{4\pi} I \int \frac{\vec{dl} \times \vec{r}}{b^3}$$

$$= \frac{\mu_0}{4\pi} I \int \frac{b^2 d\theta}{b^3}$$

$$= \frac{\mu_0 I}{4\pi} \times \frac{\pi}{b} = \frac{\mu_0 I}{4b} \hat{k}$$

$$\& \ \vec{B}_{in} = -\frac{\mu_0 I}{4a} \hat{k}$$

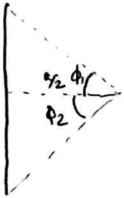
$$\therefore \ \vec{B}_{net} = -\frac{\mu_0 I}{4}\left(\frac{1}{a} - \frac{1}{b}\right) \hat{k}$$

Problem 2

a) Determine the magnetic field at the center of a square loop of wire carrying current I. Let the length of the sides be 'a'. Find the magnetic field at the origin.

b) Use symmetry to find the z component of the magnetic field at a corner of the loop.

a) Magnetic field due to a finite wire.

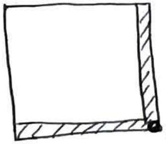


$$\vec{B} = \frac{\mu_0 I}{4\pi(a/2)}(\sin\phi_1 + \sin\phi_2)$$

$$\therefore \text{ Net } B_z = 4\frac{\mu_0 I}{4\pi(a/2)} \quad 2 \times \frac{1}{\sqrt{2}}$$

$$= 8\sqrt{2}\frac{\mu_0 I}{4\pi a}$$

b)

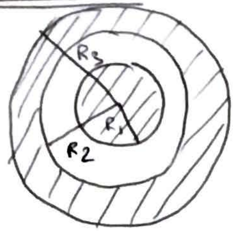


← No contribution from shaded lines

$\therefore$ Consider a square of side $2a$ and take $1/4$ of the contribution

$$\frac{1}{4} \times 8\sqrt{2}\frac{\mu_0 I}{4\pi(2a)} = \sqrt{2}\frac{\mu_0 I}{4\pi a}$$

Problem 3



$\odot \hat{k}$

Two part conductor consisting of a solid inner conducting cylinder of radius R_1 separated by an insulator from an outer conducting cylinder with inner radius R_2 & outer radius R_3.

Here $R_2 = \sqrt{R_3^2 - R_1^2}$.

$\vec{J}_{in} = J\hat{k}$

$\vec{J}_{out} = -J\hat{k}$

Assume a long cylinder and ignore edge effects.

a) Find the magnetic field B everywhere

b) Find vector potential A everywhere.

a) Using symmetry arguments

$$\vec{B} = B(r)\,\hat{\theta}$$

As $\oint B.ds = \mu_0 I_{enc} \Rightarrow B = \dfrac{\mu_0 I_{enc}}{2\pi r}$

$$I_{enc} = \begin{cases} \pi r^2 J & , r < R_1 \\ \pi R_1^2 J & , R_1 < r < R_2 \\ \pi R_1^2 J - \pi(r^2 - R_2^2)J & , R_2 < r < R_3 \\ \pi R_1^2 J - \pi(R_3^2 - R_2^2)J & , r > R_3 \end{cases}$$

using $R_2 = \sqrt{R_3^2 - R_1^2}$

$$\Rightarrow I_{enc} = \begin{cases} \pi r^2 J, & r < R_1 \\ \pi R_1^2 J, & R_1 < r < R_2 \\ \pi(R_3^2 - r^2)J, & R_2 < r < R_3 \\ 0 & , r > R_3 \end{cases}$$

$$\therefore B(r) = \begin{cases} \mu_0 r J/2 & , r < R_1 \\ \mu_0 R_1^2 J/2r, & R_1 < r < R_2 \\ \mu_0 (R_3^2 - r^2)J/2r, & R_2 < r < R_3 \\ 0 & , r > R_3 \end{cases}$$

b) We work in Coulomb Gauge where $(\nabla \cdot A = 0)$. Here, the vector potential is parallel to the current. $\therefore A = A_0 \hat{z}$

Also

$$\vec{B} = \vec{\nabla} \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta}$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{z} = B(r)\,\hat{\theta}$$

By equating coefficients,

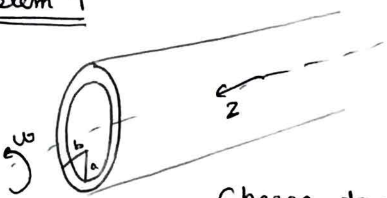
$$\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = B(r)$$

As $A_r = 0$

$$A_z = -\int B(r)\, dr.$$

$$A_z = \begin{cases} \mu_0 r^2 J/4, & r < R_1 \\ \mu_0 \frac{R_1^2 J}{2} \ln r, & R_1 < r < R_2 \\ \frac{\mu_0}{4}(2R_3^2 \ln r - r^2)J, & R_2 < r < R_3 \\ 0, & r > R_3 \end{cases}$$

Problem 4



Charge density ρ

a> Show that volume current density in cylinder $J(r) = \alpha r$
Find α in term of ρ & ω.

b> Magnetic field in $r < a$, $a < r < b$ and $r > b$.

c> Find $\vec{A}(r)$ Hint: use stoke's theorem

a> $\vec{J} = \rho \vec{v}$

$|J| = \rho r \omega$

$\therefore \alpha = \rho \omega$

b> $dI = \rho \omega r \, dr$

$r > b$

$B = 0$ (same analysis as solenoids

$a < r < b$

$$\vec{B}_I = \frac{\int_r^b \mu_0 \rho \omega r \, dr \, \cancel{l}}{\cancel{l}} \hat{z}$$

$$= \frac{\mu_0 \rho \omega}{2} (b^2 - r^2) \hat{z}$$

$r < a$

$$\vec{B}_{II} = \int_a^b \mu_0 \rho \omega r \, dr \hat{z} = \frac{\mu_0 \rho \omega}{2} (b^2 - a^2) \hat{z}$$

c) $\oint \vec{A}\cdot d\vec{l} = \int(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int \vec{B}\, d\vec{a}$

$$\oint \vec{A}\cdot d\vec{l} = A\, 2\pi r$$

$r < a$

$$A\, 2\pi r = \int \vec{B}\cdot da = B_{11}\pi r^2$$

$$\Rightarrow \vec{A} = \frac{B_{11} r}{2}\hat{\theta} = \frac{\mu_0 \rho \omega}{4} r\,(b^2 - a^2)\,\hat{\theta}$$

$a < r < b$

$$A\, 2\pi r = B_{11}\pi a^2 + \int_a^r B_1\, 2\pi r\, dr$$

$$\Rightarrow \vec{A} = \left(\frac{\mu_0 \rho \omega}{2}\frac{(b^2-a^2)\cancel{\pi} a^2}{2\cancel{\pi} r} + \frac{\int_a^r \frac{\mu_0\rho\omega}{2}(b^2-r^2)\,\cancel{2\pi} r\, dr}{\cancel{2\pi} r}\right)\hat{\theta}$$

$$= \left(\frac{\mu_0\rho\omega}{4r}(b^2-a^2)a^2 + \frac{\frac{\mu_0\rho\omega}{4}b^2(r^2-a^2) - \frac{\mu_0\rho\omega}{8}(r^4-a^4)}{r}\right)\hat{\theta}$$

$$= \frac{\mu_0\rho\omega}{8r}\left(-a^4 - r^4 + 2b^2r^2\right)\hat{\theta}$$

$r > b$

$$A\, 2\pi r = B_{11}\pi a^2 + \int_a^b B_1\, 2\pi r\, dr$$

$$\Rightarrow \vec{A} = \frac{\mu_0\rho\omega}{8r}\left(-a^4 + b^4\right)\hat{\theta}$$