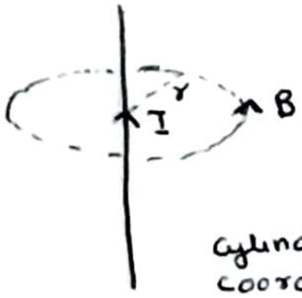
Lecture 2

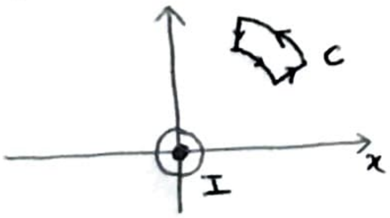
Experimentally, we can find out,



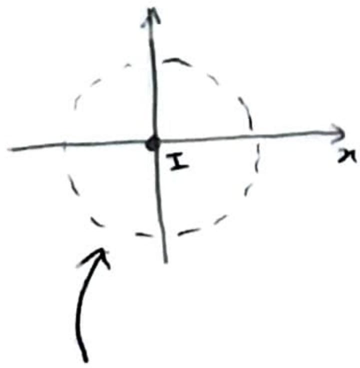
$$\vec{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\phi}$$

cylindrical coordinate system

Using this result, Dave showed in class that



$\oint \vec{B}\cdot d\vec{l} = 0$ when the current doesn't enclose a line current.



$\oint \vec{B}\cdot dl = \mu_0 I$ when current is enclosed.

where $\mu_0 = 4\pi \times 10^{-7}$ H/m or T m/A

Permeability of free space

[Comment on how this can be any shape by adding contours that contribute zero]

Note :- Our equation in general here assume time independent current i.e. charges in motion with constant velocity. Accelerating particles have a different physics but we will assume for now that ~~[illegible]~~ Ampere law holds even with bent wire

Now,

$$\oint_C \vec{B}\, dl = \mu_0 I$$

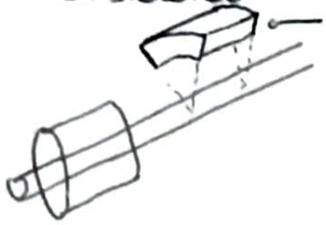
$$\Rightarrow \oint_C \vec{B}\cdot dl = \mu_0 \int \vec{J}\cdot d\vec{a}$$

(using stoke's law

$$\boxed{\Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$ ← one of the Maxwell equations

Qualitative description of divergence of $\vec{B}$

We discuss for a straight line current. All other configurations can be considered to be superposition of several straight line currents



— whatever enters leaves from the other side

$\vec{B}$ has a $\hat{\theta}$ component only

$\vec{\nabla} \cdot \vec{B} = 0$ ← implies that we do not find magnetic monopoles freely in nature. They exist as dipoles.

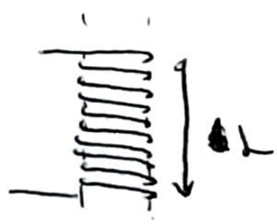
Theorem 6.1 in Purcell and Morin discusses the uniqueness of $\vec{B}$ when $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ and $\vec{\nabla} \cdot B = 0$.

To Review

Electrostatics	Magnetostatics
$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$	$\vec{\nabla} \cdot \vec{B} = 0$
$\vec{\nabla} \times \vec{E} = 0$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Problem 1

Find the magnetic field of a very long solenoid consisting of N closely wound turns of radius R and carrying a current I.



We consider that the wires are nearly horizontal.

B_r → must be zero otherwise $\vec{\nabla} \cdot \vec{B}$ won't be zero.

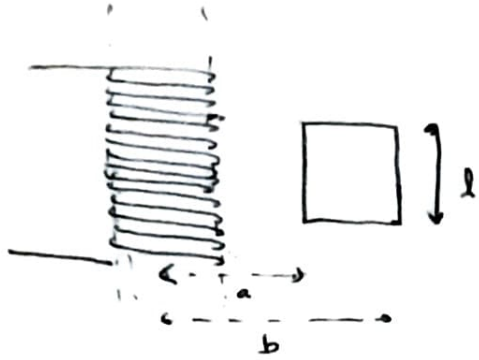
The dot product of magnetic field ⊥ to the top and bottom surfaces with 'da' must cancel as the magnetic field both at the upper and lower surface must be in the same direction.

B_θ →

I_{enc} by amperian loop is zero.

$B_\theta = 0$

$B_z \rightarrow$



Let us consider $B_z(x)$

$$\oint B_z . dl = \mu_0 I_{enc}$$

$$\Rightarrow B_z(b) - B_z(a) = 0$$

Now for $b = \infty$, $B_z(\infty)$ must be zero for $\vec{B}$ to be physical.

$\therefore B_z(a) = 0$ for arbitrary a, $a > R$

$$\oint \vec{B} . \vec{dl} = \mu_0 I_{enc}$$

$$\Rightarrow B_z(r)\, l = \mu_0 n i\, l \qquad (n = N/L)$$

$$\Rightarrow B_z(r) = \mu_0 n i$$

Vector Potential

In electrostatics, we had $\vec{\nabla} \times \vec{E} = 0$ and we could express

$$\vec{E} = -\vec{\nabla}\phi$$

Similarly, in magnetostatics, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ can be used to describe $\vec{B}$ in terms of a scalar potential only when $\vec{J} = 0$.

Instead, $\vec{\nabla} . \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ as divergence of curl is always zero.

Again just like in electrostatics where we use

$\oint \vec{E} . dl = 0 \Rightarrow \int_c \vec{E}\, \vec{dl}$ is path independent

Similarly, $\oint \vec{B}\, \vec{da} = 0 \Rightarrow \int_C \vec{B} . \vec{da}$ is dependent on the contour defining the surface
(from $\nabla . B = 0$)

Gauge Invariance

Say $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{A}' = \vec{A} + \nabla \chi'$$

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla} \chi) = \vec{\nabla} \times \vec{A} = \vec{B}$$

$$\left.\begin{aligned} \vec{A} &\rightarrow \vec{A} + \nabla \chi \\ \phi &\rightarrow \phi + c \end{aligned}\right\} \text{gauge invariance}$$

Further,

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 J$$

we choose the Coulomb gauge to impose $\vec{\nabla} \cdot \vec{A} = 0$

$$\Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Using Green's function,

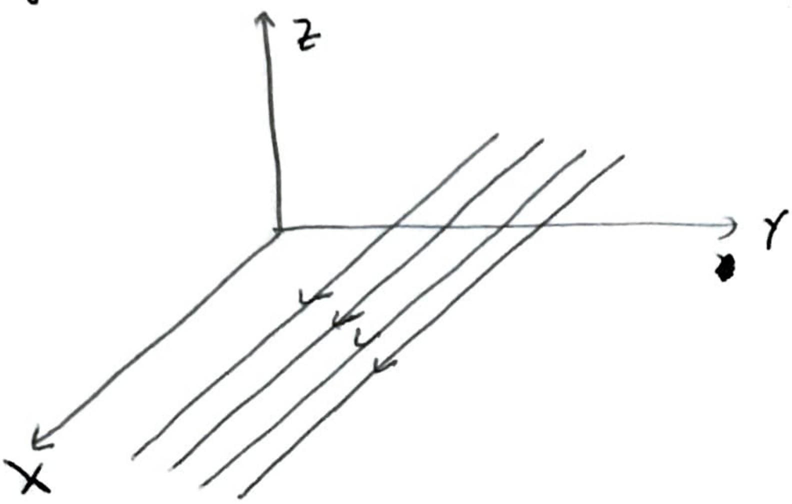
$$\vec{A}(r_1) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r_2)}{|\vec{r}_2 - \vec{r}_1|} dV(r_2)$$

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times (\vec{r}_1 - \vec{r}_2)}{(\vec{r}_1 - \vec{r}_2)^3}$$

Line Current	Surface Current
$\vec{A}(r_1) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r_2)\, dV(r_2)}{\lvert\vec{r}_2 - \vec{r}_1\rvert}$	$A(r_1) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}\, da(r_2)}{\lvert\vec{r}_2 - \vec{r}_1\rvert}$
$\vec{B}(r_1) = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times (\vec{r}_1 - \vec{r}_2)}{\lvert\vec{r}_1 - \vec{r}_2\rvert^3} dV(r_2)$	$B(r_1) = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times (\vec{r}_1 - \vec{r}_2)\, da(r_2)}{\lvert\vec{r}_1 - \vec{r}_2\rvert^3}$
$= \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times (\vec{r}_1 - \vec{r}_2)}{\lvert\vec{r}_1 - \vec{r}_2\rvert^3}$	

Problem 2

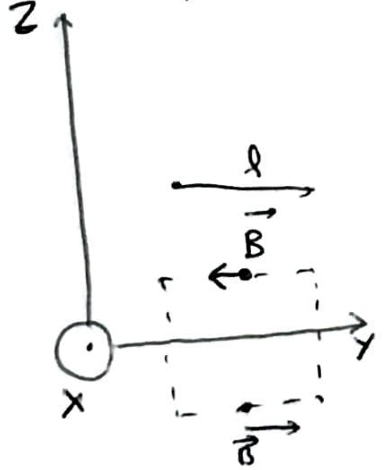
Find the magnetic field of an infinite uniform surface current $\vec{K} = K\hat{x}$ flowing through the x-y plane



$B_x \rightarrow$ cannot be non zero from Biot Savart's law

$B_z \rightarrow$ cannot be non zero. Contribution by wire segment to the left (along y) cancel out the contributions by wire segments to the right (along y.

$B_y \rightarrow$ Now we can quickly implement ampere's law.



$$\oint \vec{B} \cdot dl = \mu_0 I_{enc}$$

$$\Rightarrow 2Bl = \mu_0 K l$$

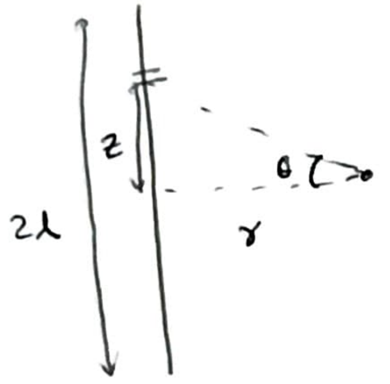
$$\Rightarrow B = \frac{\mu_0 K}{2}\hat{y} \quad \text{for } z < 0$$

$$-\frac{\mu_0 K}{2}\hat{y} \quad \text{for } z > 0$$

Problem 3

Find the vector potential of a thin infinite wire.

(I am doing only to show how to use it. This is hardly ever asked of you)



$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_{-l}^{l} \frac{d\vec{l}}{|\vec{z} - \vec{r}|}$$

$$= \frac{\mu_0 I}{2\pi} \hat{z} \int_0^{l} \frac{dz}{(z^2 + r^2)^{1/2}}$$

$$= \frac{\mu_0 I}{2\pi} \ln \left(\frac{\sec\alpha + \tan\alpha}{\sec(-\alpha) + \tan(-\alpha)} \right) \hat{z}$$

$$= \frac{\mu_0 I}{2\pi} \ln \left(\frac{\sqrt{R^2 + l^2} + l}{R} \right) \hat{z}$$

As $l \to \infty$

$$= \frac{\mu_0 I}{2\pi} \ln \left(2l/R \right) \quad [\text{goes to } \infty]$$

On the other hand, Purcell derives,

$$\vec{A} = -\frac{\mu_0 I}{2\pi} \ln(r)\, \hat{z}$$

Both answers are correct as they are related by a gauge transformation

$$\vec{\nabla} f = \frac{\mu_0 I}{2\pi} \ln(2l)$$

Similarly, for inside the wire

arbitrary constant

$$\vec{A} = \frac{-\mu_0 I}{4\pi R^2} (r^2 - b^2)\, \hat{z}$$

When solving both inside and outside the cylinder, choose 'b' to impose continuity.

Discontinuous $\vec{A}$ would imply infinite magnetic field i.e. magnetic monopoles in the region.

How would you find $\vec{B}$ from the above?

Note :- It is always advisable to use the general result to find $\vec{B}$ and then impose limits.