Lecture 17

While dielectric responses of materials are quite strong, the magnetic responses are comparatively much weaker and slightly more complex.

We will quickly list all the results we showed for electric dipoles in the case of magnetic dipoles.

Primarily 3 types of responses

→ Diamagnetism – feeble response, repulsive
→ Paramagnetism – stronger in certain materials, attractive
→ Ferromagnetism – very strong, attractive.

Lets start by discussing the nature of a magnetic dipole without going into which of these 3 types it belongs to.

For localized charge distribution

$$\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\, dv'}{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho\, dv' + \frac{1}{r^2}\int \rho r' \cos\theta\, dv' + \dots\right]$$

$$\phi_{dipole} = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

For localized current distribution.

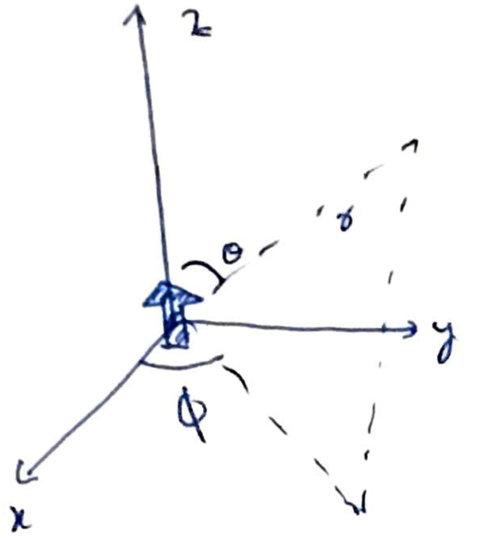
$$A = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l}'}{r}$$ ← note that this is a vector

$$= \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\underbrace{\int d\vec{l}'}_{\text{always zero for loop}} + \frac{1}{r^2}\int r'\cos\theta\, d\vec{l}' + \dots\right]$$

$$A_{dipole} = \frac{\mu_0 I}{4\pi}\int \frac{\vec{a}\times\hat{r}}{r^2}$$

$$= \frac{\mu_0\, \vec{m}\times\hat{r}}{4\pi r^2}$$

Here $\vec{m} = I\vec{a}$ (Note the direction)



$$\vec{A} = \frac{\mu_0}{4\pi} \frac{|m| \sin\theta}{r^2} \hat{\phi}$$

$$B_r = \frac{\mu_0 |m|}{2\pi r^3} \cos\theta$$

$$B_\theta = \frac{\mu_0 |m| \sin\theta}{4\pi r^3}$$

$$B_\phi = 0$$

Further,

$$\tau = \vec{m} \times \vec{B}$$

$$\text{Energy} = -\vec{m} \cdot \vec{B}$$

$$\text{Force} \left(\text{in non-uniform field}\right) = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

$$= (\vec{m} \cdot \vec{\nabla})\vec{B} + (\vec{m} \times (\vec{\nabla} \times \vec{B}))$$

In electric dipole

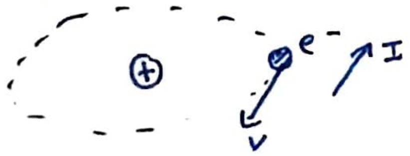
$$\tau = \vec{p} \times \vec{E}$$

$$E = -\vec{p} \cdot \vec{E}$$

$$\text{Force} = (\vec{p} \cdot \vec{\nabla})\vec{E}$$

Magnetic Polarizability

Unlike non-polar (electric) atoms, all atoms have an associated magnetic dipole.



$$I = \frac{ev}{2\pi r}$$

$$\therefore m = \frac{evr}{2}$$

Now, I subject it to an external magnetic field that increases from 0 to B_0

$$\oint \vec{E} \cdot \vec{dl} = \cdot \frac{\partial \phi_B}{\partial t} \quad \text{(faradday's law)}$$

$$\Rightarrow E = \frac{r}{2} \frac{\partial B}{\partial t}$$

$$\therefore m\frac{dv}{dt} = qE \Rightarrow \Delta v = \frac{qr}{2m} B_0$$

$$\Delta m = \frac{q r \Delta v}{2} = \frac{q^2 r^2}{4m} B_0$$

$\underbrace{}$ → similar to electric polarizability.

Just like electric polarizability,

$$\frac{\mu_0}{4\pi}\frac{\Delta m}{B_0} = \frac{\mu_0}{4\pi}\frac{q^2 r^2}{2m} = \frac{r^2}{4}\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mc^2}\right)$$

$$= \underbrace{r^3}_{\text{Volume of atom}} \underbrace{K}_{\text{some numerical factor}}$$

We have assumed here that radius doesnt change in all this.
Lets just plug that hole.

$$\text{Centripetal Force} = \frac{M(V_0+\Delta v)^2}{r} \simeq \frac{MV_0^2}{r} + \frac{2MV_0\Delta V}{r}$$

$$\text{Force provided by magnetic field} = q(V_0+\Delta V)B$$

$$= (V_0+\Delta V)\frac{2m\Delta V}{r}$$

$$\simeq \frac{2mV_0\Delta V}{r}$$

∴ radius remains the same upto a first order in ΔV.

Problem 1

A disk with radius R has uniform charge density σ and spins with angular speed ω. Find the magnetic dipole moment as measured far away at some radius r_0.

$$dI(r) = \sigma\, 2\pi r\, dr \, \frac{v}{2\pi r} = \sigma v\, dr$$

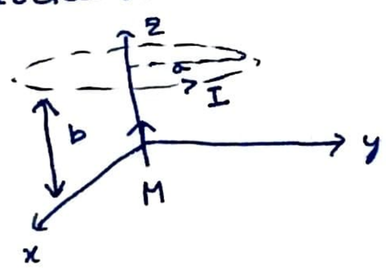
$$dm = dI\, \pi r^2$$

$$= \sigma \omega \pi r^3 dr$$

$$\therefore\ m = \frac{\sigma \omega \pi R^4}{4}$$

Problem 2

Consider a magnetic dipole $\vec{M} = m_0 \hat{z}$ located at the origin. A distance 'b' above the dipole is a ring of radius 'a' carrying current I counterclockwise.



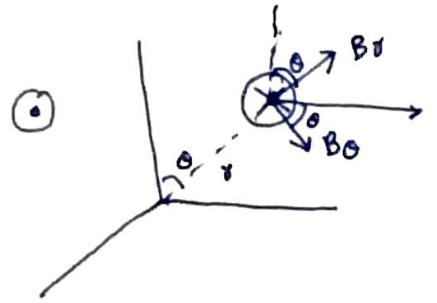
a) Find F_{net} on the loop.

b) Now approximate the loop as a dipole and calculate the force on it again

c) Find the relative error in the above results.

d) Where does the error drop to 1%

a)



$$B_H = B_r \sin\theta + B_\theta \cos\theta$$

$$= \frac{\mu_0 M_0}{4\pi r^3}\left[2\cos\theta \sin\theta + \sin\theta\cos\theta\right]$$

$$= \frac{\mu_0 M_0}{4\pi r^3}\,\frac{3}{2}\sin(2\theta)$$

$$F_z = I\, \cancel{2\pi a}\, \frac{\mu_0 M_0}{4\cancel{\pi} r^3}\,\frac{3}{\cancel{2}}\sin 2\theta\,(-\hat{z})$$

$$= \frac{3 I \mu_0 M_0}{4}\,\frac{a}{(a^2+b^2)^{3/2}}\sin 2\theta\,(-\hat{z})$$

$$= \frac{3I}{2}\mu_0 M_0\,\frac{a^2 b}{(a^2+b^2)^{5/2}}(-\hat{z})$$

b) $\vec{M}_{loop} = \pi a^2 I\,\hat{z}$

$$F = (\vec{m}\cdot\vec{\nabla})\vec{B}\quad \hat{z}$$

$$= \pi a^2 I \frac{\partial}{\partial z}(\vec{B})$$

$$= \pi a^2 I \frac{\partial}{\partial z}\left(\frac{\mu_0 M_0}{2\pi z^3}\right)\hat{z}$$

$$= -\frac{3\mu_0 M_0 a^2 I}{2 z^4}\hat{z}$$

$$= -\frac{3\mu_0 M_0 a^2 I}{2 b^4}\hat{z}$$

c) $$\eta = \left|\frac{F_{real} - F_{approx}}{F_{approx}}\right|$$

$$= \left|1 - \frac{F_{approx}}{F_{real}}\right|$$

$$= \left|1 - \frac{(a^2+b^2)^{5/2}}{b^5}\right|$$

$$= \left|\left(1 + \left(\frac{a}{b}\right)^2\right)^{5/2} - 1\right|$$

d) For $b \gg a$

$$\eta = \left|1 + \frac{5}{2}\left(\frac{a}{b}\right)^2 - 1\right|$$

For $\eta = 0.01$, $b = a\sqrt{\frac{500}{2}} \approx 16a$