Lecture 16

For this class, we will avoid dwelving into theory in detail. This is primarily because we would like to solve a number of problems insted. I will list a few things to remember.

If all the space in which electric field exists is occupied by a dielectric, the $\vec{E}_{net} = \dfrac{\vec{E}_{vac}}{K}$

This can be obtained for the capacitors we studied in the last class,

$$\vec{E}_{net} = \vec{E}_{vac} + \vec{E}_{ind}$$

$$\Rightarrow \vec{E}_{net} = \vec{E}_{vac} - \frac{\vec{P}}{\varepsilon_0} \quad (\text{using } \vec{P} = \varepsilon_0 \chi_e \vec{E}_{net})$$

$$\Rightarrow \vec{E}_{net} = \vec{E}_{vac} - \frac{\varepsilon_0 (K-1) \vec{E}_{net}}{\varepsilon_0}$$

$$\Rightarrow \vec{E}_{net} = \frac{\vec{E}_{vac}}{K}$$

However, if there exists a $\vec{E}$ outside a dielectric, the expression is usually more involved.

Of these, the most common example is a polarized dielectric sphere

$\sigma = P\cos\theta$

which can be approximated by two oppositely charged spheres that are slightly displaced.

Consider N ~~atom~~ molecules per unit volume,

$$P_0 = Qs = \frac{4}{3}\pi r_0^3 N q s = \frac{4\pi}{3} r_0^3 P$$

ϕ & $\vec{E}$ outside the sphere can be approximated by a dipole at the center (P_0)

$$\therefore \phi \text{ at the boundary} = \frac{P_0 \cos\theta}{4\pi\varepsilon_0 r^2} = \frac{P r_0 \cos\theta}{3\varepsilon_0} = \frac{Pz}{3\varepsilon_0}$$

Using uniqueness theorem, $E_z = -P/3\varepsilon_0$

$$\therefore \vec{E}_{net}^{\,(inside)} = \vec{E}_{ex} + \vec{E}_{ind}$$

$$\vec{E}_{net} = \vec{E}_{ex} - \frac{\vec{P}}{3\varepsilon_0}$$

$$\Rightarrow \vec{E}_{net} = \vec{E}_{ex} - \frac{\varepsilon_0 (K-1)\vec{E}_{net}}{3\varepsilon_0}$$

$$\Rightarrow \vec{E}_{net} = \frac{3}{2+K}\vec{E}_{ex}$$

$$\& \ \vec{P} = 3\frac{(K-1)}{K+2}\varepsilon_0 E_0$$

[For dielectrics with surface charge densities, $E_\perp$ is discontinuous & $E_\parallel$ is continuous]

Now, consider placing a charge Q inside a dielectric,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 K r^2}$$

here, $\int K\vec{E}\cdot d\vec{a} = \frac{Q}{\varepsilon_0}$ ← free charge (not decided by the material)

$$\Rightarrow \vec{\nabla}\cdot(K\vec{E}) = \frac{\rho_{free}}{\varepsilon_0}$$

on the other hand,

$$\int \vec{E}\cdot d\vec{a} = \frac{Q_{net}}{\varepsilon_0}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}(\rho_{free} + \rho_{bound})$$

$$\therefore \vec{\nabla}\cdot(\underbrace{(K-1)\vec{E}}_{\vec{P}/\varepsilon_0}) = -\rho_{bound}/\varepsilon_0$$

$\therefore$ We can define $\vec{D}$ (displacement field) $= \varepsilon_0\vec{E} + \vec{P}$

$$\therefore \vec{\nabla}\cdot\vec{D} = \rho_{free}$$

$\therefore \vec{D} = K\varepsilon_0\vec{E} = \varepsilon\vec{E}$

ε ↳ relative permeability.

Also $\vec{\nabla}\cdot\vec{P} = -\rho_{bound}$ & for a surface $\vec{P}\cdot\hat{n} = \sigma_b$

[For D, $D_\perp$ is continuous & $D_1^{\parallel} - D_2^{\parallel} = P_1^{\parallel} - P_2^{\parallel}$]

Physical explanation of dielectric constant,

$$\vec{P} = N\vec{p} = N\alpha\vec{E} \qquad \text{(for non polar molecules)}$$

& $$\chi_e = \frac{P}{\varepsilon_0 E} = \frac{N\alpha}{\varepsilon_0}$$

$$\therefore \quad k = 1 + \frac{N\alpha}{\varepsilon_0}$$

$$= 1 + 4\pi \underbrace{N}_{10^{25}} \frac{\alpha}{4\pi\varepsilon_0} \approx 10^{-30}$$

$$= 1 + 0.000\ldots$$
some value

for polar molecules, it is about aligning the dipoles in the direction of the electric field.

$$\vec{P} \approx Np \text{ (fraction of extra molecules aligned forward)}$$

$$\approx Np\left(\frac{pE}{KT}\right) = \frac{Np^2}{KT} E$$

$$\therefore \chi_e \approx \frac{Np^2}{\varepsilon_0 KT} \qquad \text{(this explain the } \chi_e \text{ upto a factor)}$$

[Discuss changing $\vec{E}$ field]

Bound Charge current

When $\vec{P}$ slightly changed due to whatever factor. Imagine shifting all positive charges a bit further by ds in time dt.

9/11/18

$$\vec{J} = \rho\vec{v} = Nq\frac{ds}{dt} = N\frac{d\vec{p}}{dt} \Rightarrow \vec{J} = d\vec{P}/dt$$

Ampere's law becomes,

$$\vec{\nabla} \times \vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\left(\vec{J}_{bound} + \vec{J}_{free}\right)$$

$$= \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\frac{\partial\vec{P}}{\partial t} + \mu_0 J_{free}$$

includes vacuum & bound displacement current

$$= \mu_0\frac{\partial D}{\partial t} + \mu_0 J_{free} = \mu_0\varepsilon\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J}_{free}$$

∴ Maxwell's Equations for dielectris with no bound & free charge/current

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad , \quad \vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon \frac{\partial \vec{E}}{\partial t} \quad , \quad \vec{\nabla} \cdot B = 0$$

Plugging them into one another as before,

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon}} \quad \& \quad E_0 = v B_0$$

∴ η (index of refraction) $= c/v = \sqrt{\frac{\mu_0 \varepsilon}{\mu_0 \varepsilon_0}} = \sqrt{K}$

η for water is 1.33 but k is ≈ 80. Can you explain?

Problem 1

Consider a spherical shell with inner radius a and outer radius b. The shell is composed of a dielectric material which is polarized such that $\vec{P} = \frac{k}{r}\hat{r}$ for some constant k.

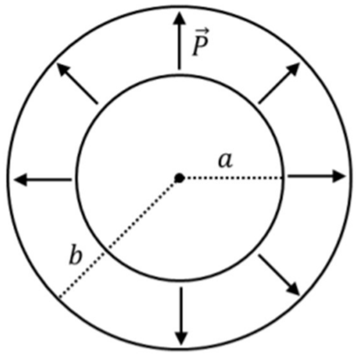
(a) Find the bound charge density and surface densities for this sytem.

(b) Does this system have free charge densities? If so, find them.

(c) What is the electric field?

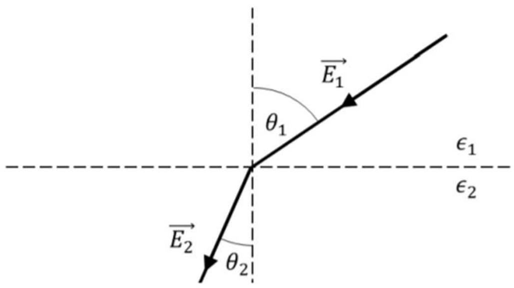
(d) Using the electric field, find the electric displacement field everywhere.

(e) Using the results of part (b), find the electric displacement field. Does this agree with your result from part (d)?



Problem 2

Consider the interface of two linear dielectrics with permittivities ϵ_1 and ϵ_2. An electric field, $\vec{E_1}$, makes an angle θ_1 with respect to the normal vector of the interface. The electric field continues into the second dielectric, now $\vec{E_2}$, making an angle θ_2 with respect to the normal vector of the interface.



(a) Find the electric displacement field and polarization vector for each dielectric.

(b) Using the boundary conditions for either the electric displacement field or the electric field, find expressions relating the fields in the two dielectrics.

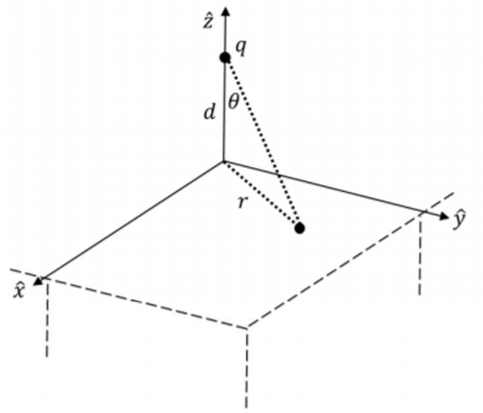
(c) By combining these two expressions, find a single equation which relates the angles θ_1 and θ_2 to the permittivities ϵ_1 and ϵ_2 without depending on the electric field strengh.

(d) Assume $\epsilon_1 < \epsilon_2$. What happens to the electric field as it passes from the first dielectric to the second? In the limit $\epsilon_2 \to \infty$ the second dielectric becomes a conductor. What happens to the electric field?

Problem 3

Consider a charge q located a distance d above a block of linear dielectric material. This dielectric block completely fills the bottom half space ($z < 0$) and extends infinitely in the x and y directions. The dielectric has a permittivity $\epsilon = \epsilon_0 (1 + \chi_e)$.

(a) Due to the point charge, the dielectric material will become polarized with polarization $\vec{P}$. What is the bound charge at the surface in terms of the electric field just below the surface of the dielectric?



(b) This bound charge induces an electric field both above and below the surface of the dielectric. What is the electric field of the bound charge just below the surface? Leave this in terms of the bound charge surface density.

(c) Consider just the electric field from the point charge q. What is the electric field just below the surface of the dielectric as a function of r? What is the component normal to the surface?

(d) What is the total electric field just below the surface of the dielectric normal to the surface? Using your results from part(a), find an expression for the bound charge as a function of r.

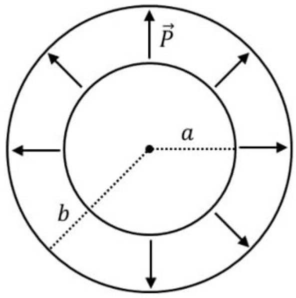
(e) What is the total induced bound charge on the surface of the dielectric, q_b? (Hint: $\int_0^\infty \frac{x}{(1+x^2)^{3/2}} dx = 1$)

(f) While we will not prove it here, we can use the method of images to treat the bound charge as being a single point charge q_b at $(0, 0, -d)$. Using the method of images, what is the force on the point charge q?

(g) In the limit $\chi_e \to \infty$, the dielectric block becomes a conductor. Check that your results for parts (d)-(f) agree with those for a conductor in this limit.

Solution 1

Consider a spherical shell with inner radius a and outer radius b. The shell is composed of a dielectric material which is polarized such that $\vec{P} = \frac{k}{r}\hat{r}$ for some constant k.



(a) The bound charge surface densities are given by $\vec{P} \cdot \hat{n}$ where $\hat{n}$ is the normal vector of the surface. Considering the inner and outer surfaces of the spherical shell, we find,

$$\sigma_{\text{inner}} = \vec{P} \cdot (-\hat{r}) = -\frac{k}{a}$$
$$\sigma_{\text{outer}} = \vec{P} \cdot \hat{r} = \frac{k}{b}$$

The bound charge density can be found by taking the divergence of $\vec{P}$. This gives,

$$\begin{aligned}\rho_{\text{bound}} &= -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right) \\ &= -\frac{k}{r^2}\end{aligned}$$

(b) No free charges were specified for this system, only the polarization vector was given. Unless explicitly added, polarized materials only have bound charges. As such, this system does not have any free charges.

(c) The polarized material does not have a net charge. Thus, $\vec{E} = 0$ for $r < a$ and $r > b$. However, for $a \leq r \leq b$, we need to be concerned with the bound charges. Using Gauss's law,

$$\begin{aligned}
\oiint \vec{E} \cdot d\vec{A} &= \iiint \frac{\rho}{\epsilon_0} dV \\
4\pi\epsilon_0 r^2 E_r &= 4\pi a^2 \sigma_{\text{inner}} + \int_a^r 4\pi (r')^2 \rho_{\text{bound}} dr' \\
4\pi\epsilon_0 r^2 E_r &= 4\pi a^2 \left(-\frac{k}{a}\right) + \int_a^r 4\pi (r')^2 \left(-\frac{k}{(r')^2}\right) dr' \\
4\pi\epsilon_0 r^2 E_r &= 4\pi ak - 4\pi k (r-a)\, dr' \\
\vec{E} &= -\frac{k}{\epsilon_0 r}\hat{r}
\end{aligned}$$

(d) The electric displacement field is given by $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$. As both $\vec{E}$ and $\vec{P}$ vanish outside of the dielectric, $\vec{D}$ also vanishes. Examining the inside of the dielectric,

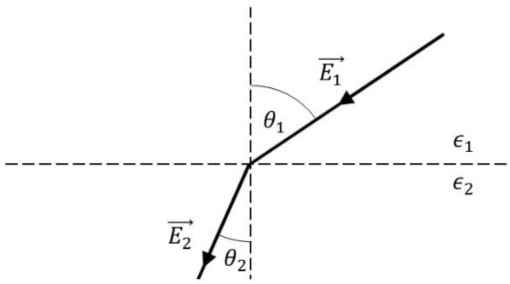
$$\begin{aligned}
\vec{D} &= \epsilon_0 \vec{E} + \vec{P} \\
\vec{D} &= \epsilon_0 \left(-\frac{k}{\epsilon_0 r}\hat{r}\right) + \frac{k}{r}\hat{r} \\
\vec{D} &= 0
\end{aligned}$$

Thus, $\vec{D} = 0$ everywhere.

(e) In part (b), we found that there are no free charges in this system. Similarly, there are no changing magnetic fields. As free charges and changing magnetic fields are what source $\vec{D}$, it follows that $\vec{D} = 0$. This is what was shown in part (d).

Solution 2

We will define the susceptibilities χ_1 and χ_2 in the usual way, $\epsilon_1 = \epsilon_0 (1 + \chi_1)$ and $\epsilon_2 = \epsilon_0 (1 + \chi_2)$.



(a) Both dielectrics are linear. Thus, we have the usual relations,

$$\vec{D}_1 = \epsilon_1 \vec{E}_1 \qquad\qquad \vec{D}_2 = \epsilon_2 \vec{E}_2$$
$$\vec{P}_1 = \epsilon_0 \chi_1 \vec{E}_1 \qquad\qquad \vec{P}_2 = \epsilon_0 \chi_2 \vec{E}_2$$

(b) We will use the boundary conditions for the electric displacement field. There are two boundary conditions which must be satisfied, $D_1^\perp - D_2^\perp = 0$ and $D_1^\parallel - D_2^\parallel = P_1^\parallel - P_2^\parallel$. Examining the boundary condition for the perpendicular components,

$$\begin{aligned} D_1^\perp &= D_2^\perp \\ \epsilon_1 \left|\vec{E}_1\right| \cos(\theta_1) &= \epsilon_2 \left|\vec{E}_2\right| \cos(\theta_2) \end{aligned}$$

Turning to the parallel components

$$\begin{aligned} D_1^\parallel - D_2^\parallel &= P_1^\parallel - P_2^\parallel \\ \epsilon_1 \left|\vec{E}_1\right| \sin(\theta_1) - \epsilon_2 \left|\vec{E}_2\right| \sin(\theta_2) &= \epsilon_0\chi_1 \left|\vec{E}_1\right| \sin(\theta_1) - \epsilon_0\chi_2 \left|\vec{E}_2\right| \sin(\theta_2) \\ \epsilon_0 \left|\vec{E}_1\right| \sin(\theta_1) &= \epsilon_0 \left|\vec{E}_2\right| \sin(\theta_2) \end{aligned}$$

(c) Dividing the boundary condition for the parallel components by the boundary condition for the perpendicular components, we find

$$\begin{aligned} \frac{\epsilon_0}{\epsilon_1} \tan(\theta_1) &= \frac{\epsilon_0}{\epsilon_2} \tan(\theta_2) \\ \frac{\tan(\theta_1)}{\tan(\theta_2)} &= \frac{\epsilon_1}{\epsilon_2} \end{aligned}$$

(d) If $\epsilon_1 < \epsilon_2$ then $\tan(\theta_1) < \tan(\theta_2)$. As θ_1 and θ_2 are bounded between 0 and $\frac{\pi}{2}$, it then follows that $\theta_1 < \theta_2$. As such, when the electric field passes into the second dielectric, it is deflected away from the normal. In the limit $\epsilon_2 \to \infty$, this effect is maximized and the outgoing electric field now travels parallel to the interface.

While it is unnatural in the context of infinite dielectric interfaces, if we have a finite dielectric block we can talk about the penetration of the electric field into the dielectric. As ϵ_2 increases, the electric field cannot reach the inner most parts of a finite dielectric block before exiting through another surface. In the limit where $\epsilon_2 \to \infty$, the electric field is completely excluded from the interior of the dielectric and only exists along the surface. This justifies the usual statement that there are no electric fields inside conductors.

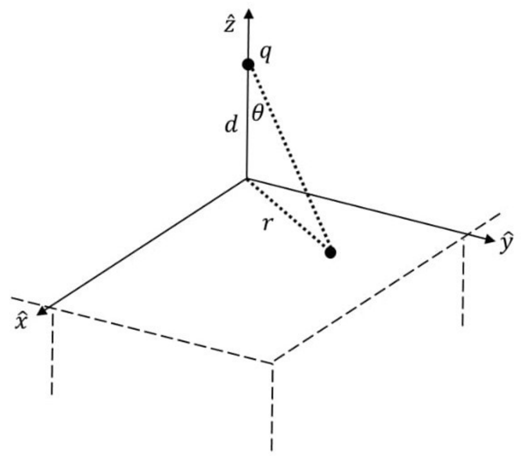
Solution 3

Consider a charge q located a distance d above a block of linear dielectric material. This dielectric block completely fills the bottom half space ($z < 0$) and extends infinitely in the x and y directions. The dielectric has a permittivity $\epsilon = \epsilon_0(1 + \chi_e)$.

(a) As the dielectric is linear, $\vec{P} = \epsilon_0\chi_e\vec{E}^{\text{below}}$. For a polarization $\vec{P}$, the bound surface charge is given by $\sigma_b = \vec{P}\cdot\hat{n}$. Thus, for the dielectric block,

$$\begin{aligned} \sigma_b &= \vec{P}\cdot\hat{z} = \epsilon_0\chi_e\vec{E}^{\text{below}}\cdot\hat{z} \\ &= \epsilon_0\chi_e E_z^{\text{below}} \end{aligned}$$

(b) To find the electric field produced by the surface charge density, we will make use of Gauss's law. We



will consider a cylindrical Gaussian surface with axis along $\hat{z}$ and cross-sectional area A. Then,

$$\begin{aligned}
\oiint \vec{E} \cdot d\vec{A} &= \iiint \frac{\rho}{\epsilon_0} dV \\
E_z^{\text{above}} A - E_z^{\text{below}} A &= \frac{\sigma_b A}{\epsilon_0} \\
-2E_z^{\text{below}} &= \frac{\sigma_b}{\epsilon_0} \\
\vec{E}^{\text{below}} &= -\frac{\sigma_b}{2\epsilon_0} \hat{z}
\end{aligned}$$

(c) Consider just the electric field from the point charge q. For a point charge, the electric field is given by, $\vec{E} = \frac{q}{4\pi\epsilon_0 R^2} \hat{R}$. Thus, the field just below the surface of the dielectric is given by, $\vec{E} = \frac{q}{4\pi\epsilon_0 (r^2+d^2)} \hat{R}$. The normal vector for the dielectric surface is $\hat{z}$. Thus,

$$E_z^{\text{below}} = -\frac{q}{4\pi\epsilon_0 (r^2 + d^2)} \cos(\theta)$$

Note that $\cos(\theta) = \frac{d}{\sqrt{r^2+d^2}}$. Thus, we may rewrite the electric field as,

$$E_z^{\text{below}} = -\frac{qd}{4\pi\epsilon_0 (r^2 + d^2)^{3/2}}$$

(d) Combining our results for the surface bound charge and the point charge, the total electric field normal to the surface just inside the dielectric is given by,

$$E_z^{\text{below}} = -\frac{\sigma_b}{2\epsilon_0} - \frac{qd}{4\pi\epsilon_0 (r^2 + d^2)^{3/2}}$$

However, from part (a), we also know that $\epsilon_0 \chi_e E_z^{\text{below}} = \sigma_b$. Eliminating the electric field, we find,

$$\frac{\sigma_b}{\epsilon_0 \chi_e} = -\frac{\sigma_b}{2\epsilon_0} - \frac{qd}{4\pi\epsilon_0 (r^2 + d^2)^{3/2}}$$

Solving for the surface charge density, we find,

$$\sigma_b = -\frac{1}{2\pi} \left(\frac{\chi_e}{2 + \chi_e} \right) \frac{qd}{(r^2 + d^2)^{3/2}}$$

(e) We can now find the total bound charge along the surface of the dielectric. Integrating over r, we find,

$$
\begin{aligned}
q_b &= \iint \sigma_b dA \\
&= \int_0^{\infty} -\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right)\frac{qd}{(r^2+d^2)^{3/2}} 2\pi r dr \\
&= -\left(\frac{\chi_e}{2+\chi_e}\right) q \int_0^{\infty} \frac{\frac{r}{d}}{\left(1+\left(\frac{r}{d}\right)^2\right)^{3/2}} \frac{dr}{d} \\
&= -\left(\frac{\chi_e}{2+\chi_e}\right) q \int_0^{\infty} \frac{x}{(1+x^2)^{3/2}} dx \\
q_b &= -\left(\frac{\chi_e}{2+\chi_e}\right) q
\end{aligned}
$$

(f) We now consider placing q_b at $(0,0,-d)$. The separation between q_b and q is then $2d$. Thus, the force on q is given by,

$$
\begin{aligned}
\vec{F} &= \frac{q_b q}{4\pi\epsilon_0 (2d)^2}\hat{z} \\
\vec{F} &= -\left(\frac{\chi_e}{2+\chi_e}\right)\frac{q^2}{4\pi\epsilon_0 (2d)^2}\hat{z}
\end{aligned}
$$

(g) Finally, we consider the limit $\chi_e \to \infty$ where the dielectric block becomes a conductor. Using our results for parts (d)-(f), we find,

$$
\begin{aligned}
\sigma_b &= -\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right)\frac{qd}{(r^2+d^2)^{3/2}} & \lim_{\chi_e\to\infty} \sigma_b &= -\frac{1}{2\pi}\frac{qd}{(r^2+d^2)^{3/2}} \\
q_b &= -\left(\frac{\chi_e}{2+\chi_e}\right) q & \lim_{\chi_e\to\infty} q_b &= -q \\
\vec{F} &= -\left(\frac{\chi_e}{2+\chi_e}\right)\frac{q^2}{4\pi\epsilon_0 (2d)^2}\hat{z} & \lim_{\chi_e\to\infty} \vec{F} &= -\frac{q^2}{4\pi\epsilon_0 (2d)^2}\hat{z}
\end{aligned}
$$

Thus, we recover the correct expressions for a conductor when we take $\chi_e \to \infty$.