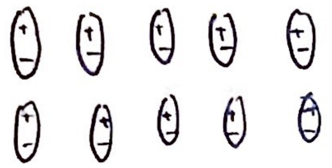
Lecture 15

In the last class we discussed about dipoles and the polarizability of atoms. Now lets consider a large number of molecules with some dipole moment $\vec{p}$ and density N. We shall ignore if this moment is permanent or induced for now.

So, in bulk let us assume that these dipoles point in the same direction

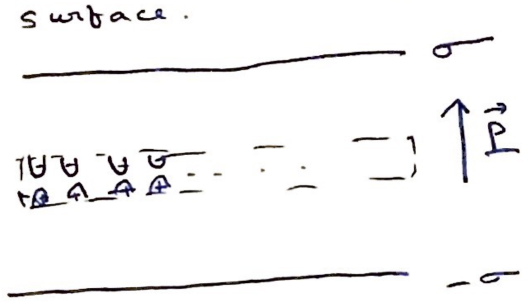


The polarization density for this configuration is given by $\vec{P} = \vec{p}N$

$\therefore$ for a small volume dv, the net dipole moment is $\vec{P}N$

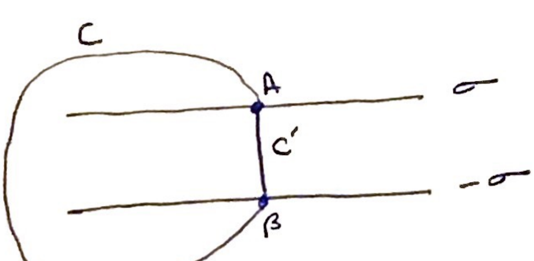
This is a reasonable approximation when N is very large and we use it to calculate for $\phi / \vec{E}$ outside the volume containing the dipoles. i.e. the dipoles are very small considering the length scales involved.

Using these considerations, we found in the class that, for a block of such molecules having polarization density $\vec{P}$, we can treat the bulk as being neutral and all charges are accumulated at the surface.



where $\sigma = \vec{P}$

But this doesnt tell us enough about the field inside the bulk. We can indeed speak about the field outside. This is because close to the molecules, fields can be in the scale of several million V/m. Instead we use the average field.



$\Delta V_C = \Delta V_{C'}$ (electrostatic fields)

& as the potential difference ΔV_C can be simply created by an $\vec{E}$ pointing downward in the bulk, we say that

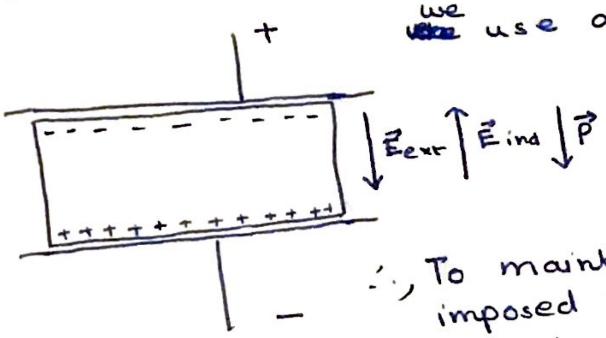
$$\langle E_{in} \rangle = \frac{\sigma}{\varepsilon_0} \text{ (downward)} = -\frac{\vec{P}}{\varepsilon_0}$$

Moving forward we shall simply refer to this as E_{in} and it should be understood as the average.

Now the question remains on how we could characterize the material involved.

To do this, we use ~~[illegible]~~ capacitors,

we use a voltage imposed by a battery.



$\therefore$ To maintain the voltage difference imposed by the battery, we need more charge on the capacitor plates.

Let this new charge be QK.
and Q' be the charge on the dielectric surface

$$\therefore \frac{QK}{A\varepsilon_0} - \frac{Q'}{A\varepsilon_0} = \frac{Q}{A\varepsilon_0}$$

(by equating voltage before and after inserting capacitor)

$$\Rightarrow \vec{E}K - \frac{\vec{P}}{\varepsilon_0} = \vec{E}$$

(This 'K' is known as the dielectric constant. The higher the value, the more dipole moment the material produces)

$$\Rightarrow \vec{P} = \vec{E}\,\underbrace{(K-1)}_{}\,\varepsilon_0$$

$\hookrightarrow \chi_e \rightarrow$ electric susceptibily)

$(K_{vac} = 1)$

Further, materials where $\vec{P} \propto \vec{E}$ are known as linear dielectrics.

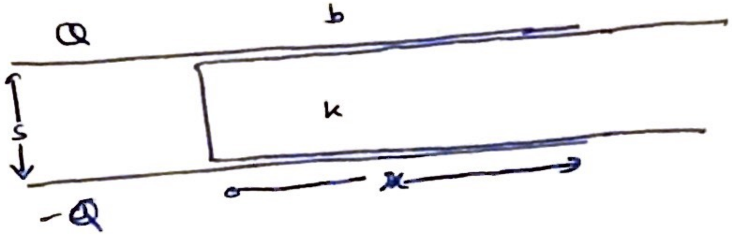
This is the simplest of scenarios as χ_e may be dependent on $\vec{E}$ itself or χ_e may be a non isotropic material.

We shall only focus on homogenous & isotropic materials alone.

For the above scenario $C' = Q'/V = \dfrac{QK}{\frac{Q}{A\varepsilon_0}d} = \dfrac{A\varepsilon_0}{d}K = CK$

Problem 1

A rectangular capacitor with side lengths a & b has separation 's' with 's' much smaller than 'a' and 'b'. It is partially filled with a dielectric with constant K. The overlap distance is x. The capacitor is isolated and has constant charge Q.



a) Find energy stored in the system?

b) Find the force on the dielectric.

a) We treat this as two parallel capacitors.

$$C_{tot} = C_1 + C_2$$

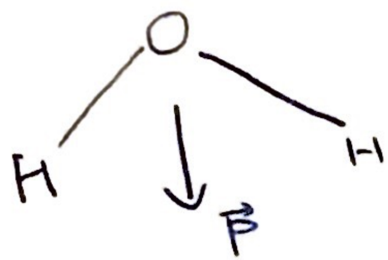
$$= \frac{\varepsilon_0 a (b-x)}{s} + \frac{\varepsilon_0 a x K}{s} = \frac{\varepsilon_0 a}{s}\left[b + (K-1)x \right]$$

$$\therefore U = \frac{Q^2}{2C} = \frac{Q^2 s}{2\varepsilon_0 a \left[b + (K-1)x \right]}$$

b) $$F = -\frac{dU}{dx} = \frac{Q^2 s (K-1)}{2\varepsilon_0 a \left[b + (K-1)x \right]}$$

Problem 2

The electric dipole moment of water molecule is 6.13×10^{-30} C-m.
Now imagine, you could make all water molecules point down. Calculate the resultant surface charge density. Find the no of extra e^-s on the surface



$$N = \frac{6 \times 10^{23}}{18\ cc}$$ ← No of molecul ← volume of mol

$$= 3.33 \times 10^{22}\ cm^{-3}$$

Now $\vec{P} = \vec{p}\, N$

$$= (6.13 \times 10^{-28}\ cm) \times (3.33 \times 10^{22}\ cm^{-3})$$

$$= 2.04 \times 10^{-5}\ C/cm^2$$

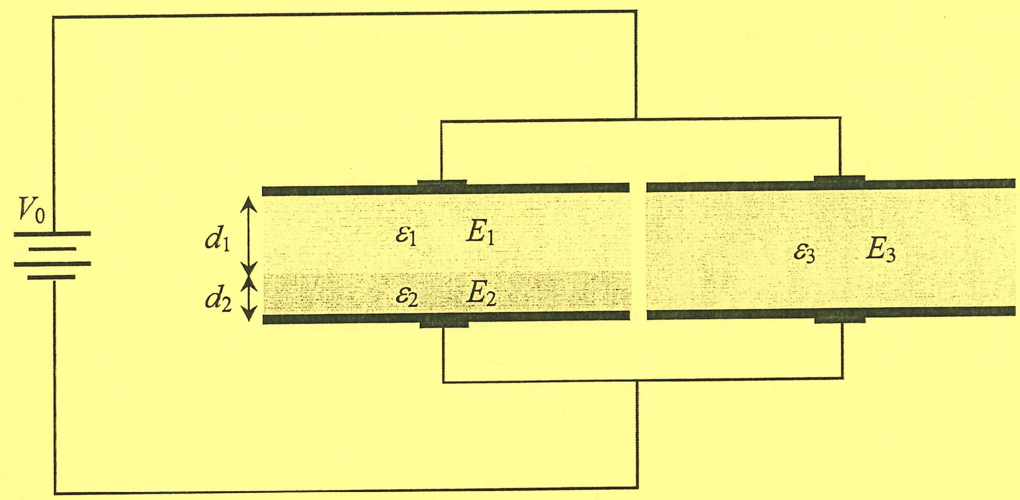
$$\sigma = \vec{P}$$

$$\therefore n_e = \frac{\sigma}{e} = 1.3 \times 10^{-14}\ cm^{-2}$$

Problem 5: Capacitor with Dielectrics

A capacitor consisting of two pairs of large parallel metal plates, each of area A and separated by a small distance $(d_1 + d_2)$, is connected to a battery of voltage V_0, as shown in the figure below. The capacitor is filled with three different kinds of dielectrics, each of area A, with dielectric constants ϵ_1, ϵ_2 and ϵ_3. As shown, the thickness of the material with dielectric constant ϵ_1 is d_1 and the thickness of the material with dielectric constant ϵ_2 is d_2.

We further assume that the length scale of the plates is much larger than their separation, so that any effects of fringe fields at the edges may be neglected.



Express your answers to parts (a) - (c) in terms of $A, d_1, d_2, V_0, \epsilon_1, \epsilon_2$ and ϵ_3.

(1 point) (a) Find the total capacitance C of the capacitor.

(2 points) (b) What are the electric fields E_1, E_2 and E_3 in the three dielectrics ?

(2 points) (c) What are the free surface charge densities σ_1 and σ_3 on each of the two top plates, above the dielectrics with ϵ_1 and ϵ_3 respectively ?

(2 points) (d) What is the bound surface charge density σ_0 on the horizontal surface between the two dielectrics with ϵ_1 and ϵ_2 ?

Problem 5: Capacitor with dielectrics

(a) (1 point) Find the total capacitance C of the capacitor.

The total capacitance C may be expressed in terms of two parallel capacitors C_{12} and C_3, where C_{12} consists of two serially connected capacitors C_1 and C_2:

$$C = C_{12} + C_3 = \frac{C_1 C_2}{C_1 + C_2} + C_3,$$

and
$$C_1 = \frac{\varepsilon_1 A}{4\pi d_1}, \qquad C_2 = \frac{\varepsilon_2 A}{4\pi d_2}, \qquad C_3 = \frac{\varepsilon_3 A}{4\pi (d_1 + d_2)}.$$

Hence, we obtain
$$C = \left[\left(\frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 d_2 + \varepsilon_2 d_1}\right) + \frac{\varepsilon_3}{d_1 + d_2}\right]\frac{A}{4\pi}.$$

(b) (2 points) What are the electric fields E_1, E_2 and E_3 in the dielectrics?

If we neglect the fringe fields, all electric fields should be considered to be perpendicular to the capacitor plates. In the absence of dielectrics, the uniform electric field is given by $E_0 = \dfrac{V_0}{(d_1 + d_2)}$. The dielectrics effectively reduce the electric fields by the corresponding dielectric constants so that $E = (E_0/\varepsilon)$. Hence, we arrive at the solutions:

$$E_1 = \frac{V_0}{\varepsilon_1 (d_1 + d_2)}, \qquad E_2 = \frac{V_0}{\varepsilon_2 (d_1 + d_2)}, \qquad E_3 = \frac{V_0}{\varepsilon_3 (d_1 + d_2)}.$$

(c) (2 points) What are the free surface charge densities σ_1 and σ_3 on each of the top plate, above the dielectrics with ε_1 and ε_3, respectively?

The voltage drop across each of the two parallel capacitors is the same, and therefore the free surface charge on each capacitor is determined by the corresponding capacitance. Thus, the free surface charge density σ_3 on the top of the capacitor containing the dielectric with ε_3 is given by the expression

$$\sigma_3 = \frac{Q_3}{A} = \frac{C_3 V_0}{A} = \frac{\varepsilon_3 V_0}{4\pi (d_1 + d_2)},$$
$$\sigma_1 = \frac{Q_1}{A} = \frac{C_{12} V_0}{A} = \left(\frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 d_2 + \varepsilon_2 d_1}\right)\frac{V_0}{4\pi}.$$

(d) (2 points) What is the surface bound charge density σ_0 on the horizontal surface between the two dielectrics with ε_1 and ε_2?

The bound surface charge density σ_0 between the two dielectrics with ε_1 and ε_2 leads the electric field discontinuity at the interface. From part (b), the discontinuity in the electric field ΔE is given by

$$\Delta E = \left(E_1 - E_2\right) = \frac{\left(\varepsilon_2 - \varepsilon_1\right)}{\varepsilon_1 \varepsilon_2 \left(d_1 + d_2\right)} V_0.$$

Therefore, the bound surface charge density at the interface is

$$\sigma_0 = \frac{\Delta E}{4\pi} = \frac{\left(\varepsilon_2 - \varepsilon_1\right)}{4\pi \varepsilon_1 \varepsilon_2 \left(d_1 + d_2\right)} V_0.$$