Lecture 14

I would like to start by saying that Dave's excellent coverage of topics has made my work today increasingly difficult. We shall therefore mostly review a few results and solve a few problems.

For a localized charge distribution $\boxed{\leftarrow \text{discuss what that means}}$

$$\phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho \, dv' + \frac{1}{r^2}\int r'\cos\theta \, dv' + \dots\dots\right]$$

$r \leftarrow$ position of point where ϕ is being measured.

$r' \leftarrow$ position of charge element

$\theta \leftarrow$ angle between r & r'

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{k_0}{r} + \frac{k_1}{r^2} + \frac{k_2}{r^3} + \dots\dots\right]$$

(multipole expansion)

$k_0 \rightarrow$ monopole moment

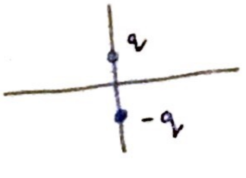
$k_1 \rightarrow$ dipole moment

$k_2 \rightarrow$ quadrapole moment.

ϕ is dominated by the first non-zero term.

The method is essentially a leading order approximation.

We shall confine our discussion to the case of dipole moment i.e. $k_0 = 0$, $k_1 \neq 0$



$$\phi = \frac{1}{4\pi\varepsilon_0 r^2}\int \hat{r}\cdot\vec{r}'\rho \, dv' = \frac{\hat{r}}{4\pi\varepsilon_0 r^2}\cdot\int \vec{r}'\rho \, dv'$$

$\therefore P = \int \vec{r}'\rho \, dv' \leftarrow$ henceforth referred to as dipole moment

$\therefore$ For our case

$$= \frac{\vec{r}}{2}q + \left(-\frac{\vec{r}}{2}\right)(-q) = \vec{r}q$$

$$\therefore \boxed{\phi = \frac{\hat{r}\cdot\vec{P}}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}$$

& Using $\vec{E} = -\vec{\nabla}\phi$

$$\boxed{E_r = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}, \quad E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}}$$

Under the influence of an electric field, force is zero, howeve

$$\tau = \vec{r}/_2 \times q\vec{E} + (-\vec{r}/_2)\times(-q\vec{E})$$

$$= \vec{r}q\times\vec{E}$$

$$\boxed{= \vec{P}\times\vec{E}}$$

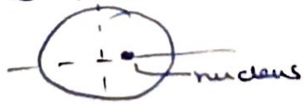
(*) Calculate work done to rotate a dipole by $180°$ when $\vec{P}$ & $\vec{E}$ points in the same direction.

$$W = \int_0^\pi \vec{\tau}\cdot d\vec{\theta}$$

$$= \int_0^\pi PE\sin\theta \, d\theta = PE(-\cos\theta)_0^\pi = 2PE$$

Under a variable $\vec{E}$, $\boxed{\vec{F} = (\vec{P}\cdot\vec{\nabla})\vec{E}}$

And, finally, the dipole moment induced in molecule under ~~the~~ a $\vec{E}$ field is



$$P = e\Delta z$$

$$= 4\pi\varepsilon_0 a^3 \vec{E}$$

We use a rule of thumb that $\vec{P} = \alpha\vec{E}$

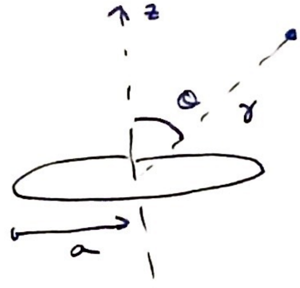
$$\therefore \alpha = 4\pi\varepsilon_0 a^3$$

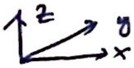
discuss the issue of ambiguity here.

(where 'a' is the typical atomic radius)

Problem 1

Consider a ring of radius 'a' with charge 'q'. A point P is located at a distand 'r' from the center of ring and angle θ wrt the axis of the ring





a) Express ϕ at P in terms of ~~moments of charge~~ increasing orders of (a/r). Use $r >> a$.

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')\,dV'}{|\vec{r}-\vec{r}'|}$$

Now $\vec{r} = (r\sin\theta, 0, r\cos\theta)$

& $\vec{r}' = (a\cos\phi, a\sin\phi, 0)$

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} \frac{(q/2\pi \cancel{a})\, \cancel{a}\, d\phi}{\sqrt{r^2 + a^2 - 2ar\cos(\phi)\sin\theta}}$$

For $r >> a$

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{q}{2\pi} \int \frac{d\phi}{r\left(1 + \left(\frac{a}{r}\right)^2 - \frac{2a\cos\phi\sin\theta}{r}\right)^{1/2}}$$

Now $\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + O(x^3)$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{2\pi r} \int d\phi \Big[1 - \frac{1}{2}\left(-2\left(\frac{a}{r}\right)\cos\phi\sin\theta + \left(\frac{a}{r}\right)^2\right) + \frac{3}{8}\left(-2\left(\frac{a}{r}\right)\cos\phi\sin\theta + \left(\frac{a}{r}\right)^2\right)^2$$

After some algebra,

$$\phi = \frac{q}{4\pi\varepsilon_0 r}\left[1 - \frac{1}{4}(3\cos^2\theta - 1)(a/r)^2 + O(a/r)^3\right]$$

b) Identify monopole, dipole & quadrapole terms.

$$\phi_{\text{monopole}} = q/4\pi\varepsilon_0 r$$

$$\phi_{\text{dipole}} = 0$$

$$\phi_{\text{quadrapole}} = -\frac{q\,a^2}{16\pi\varepsilon_0 r^3}(3\cos^2\theta - 1)$$

c) Estimate where the quadrapole term is 1% of the monopole term. (Use $\theta = 0$)

$$\left|\frac{\phi_{\text{quadrupole}}(r_*, 0)}{\phi_{\text{monopole}}(r_*, 0)}\right| = \frac{1}{100}$$

$$\Rightarrow \frac{q a^2}{8\pi\varepsilon_0 r_*^3}\left(\frac{q}{4\pi\varepsilon_0 r_*}\right)^{-1} = \frac{1}{100}$$

$$\Rightarrow r_* = 5\sqrt{2}\,a$$

$$= 7a$$

Problem 2

According to QM, ~~the~~ the e^- cloud for H atom (ground state) has density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}.$$

$q \rightarrow$ charge of e^-
$a \rightarrow$ bohr radius.

Find atomic polarizability of the atom.

$|\vec{E}|$ at radius r

$$|\vec{E}| = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = \frac{\int \rho \, dV}{4\pi\varepsilon_0 r^2} = \frac{\int \left(\frac{e}{\pi a^3}\right) e^{-2r'/a} \, 4\pi r'^2 dr'}{4\pi\varepsilon_0 r^2}$$

$$= \frac{q}{\varepsilon_0 \pi a^3 r^2} \int e^{-2r'/a} r'^2 dr$$

$$= \frac{q}{\varepsilon_0 \pi a^3 r^2} \left[-\frac{a}{2} e^{-2r'/a} \left(r'^2 + ar' + \frac{a^2}{2} \right) \right] \Bigg|_0^r$$

$$= \frac{q}{4\pi\varepsilon_0 r^2} \left[1 - \frac{\frac{a}{2} e^{-2r/a} \left(r^2 + ar + \frac{a^2}{2} \right)}{a^3/4} \right]$$

$$= \frac{q}{4\pi\varepsilon_0 r^2} \left[1 - e^{-2r/a} \left(1 + 2\frac{r}{a} + 2\left(\frac{r}{a}\right)^2 \right) \right]$$

expanding exponential

$$= \frac{q}{4\pi\varepsilon_0 r^2} \left[\frac{4}{3} \frac{r^3}{a^3} \right]$$

$$= \frac{qr}{3\pi\varepsilon_0 a^3}$$ (at equilibrium this equals the external field)

$$\therefore \alpha = 3\pi\varepsilon_0 a^3$$

(Unfortunately, the classical uniform description is closer to the experimental value)

Problem 3

$P = \alpha E$ is not fundamental relation.

Suppose, that $\rho \alpha r$ (upto R). What is the relation of P to E?

As, $\rho = Ar$

$$E = \frac{\int \rho \, dV/\varepsilon_0}{4\pi r^2} = \frac{\int A r^3 \, dr \, 4\pi}{4\pi r^2 \varepsilon_0} = \frac{A r^{\cancel{4}\,2}}{4 \cancel{\pi r^2} \varepsilon_0} = \frac{A r^2}{4 \varepsilon_0}$$

When internal field balances the external field,

$$E = \frac{ad^2}{4\varepsilon_0}$$

$$\Rightarrow d = \sqrt{\frac{4 \cdot \varepsilon_0 E}{A}}$$

$\therefore$ Induced dipole $= ed = 2e\sqrt{\frac{\varepsilon_0}{A}} \sqrt{E}$