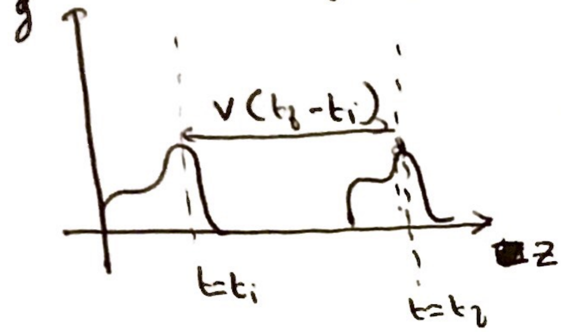
Lecture - 13

General equation of wave

$$g = f_1(z - vt) + f_2(z + vt)$$



(only for f_1)

Now for any equation of the form $g = f_1(x - vt)$

$$\cancel{\frac{\partial g}{\partial z} = \frac{\partial f}{\partial (z-vt)} \frac{d(z-vt)}{dz}}$$

$$\frac{\partial g}{\partial z} = \frac{df}{d\underbrace{(z-vt)}_{u}} \frac{\partial (z-vt)}{\partial z} = \frac{df}{du} \Rightarrow \frac{\partial^2 g}{\partial z^2} = \frac{d^2 f}{du^2}$$

$$\& \quad \frac{\partial g}{\partial t} = -v \frac{df}{du} \Rightarrow \frac{\partial^2 g}{\partial t^2} = v^2 \frac{d^2 f}{du^2}$$

$$\therefore \frac{\partial^2 g}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 g}{\partial t^2}$$ (where v is velocity)

$\therefore$ For EM fields, (in free space)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow -\nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$\Rightarrow \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$\therefore$ E must have the form of $f(x - vt)$

$\vec{B}$ follows similarly,

$$\& \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$$

Note: I have not assumed any particular form for the wave.

∴ EM waves

1) always travel with speed 'c'

2) at every point $|\vec{E}| = c\,|\vec{B}|$ (Using Faradays Law)

3) they are $\perp$ to each other and to the direction of propagation ← follows from ($\vec{\nabla}\cdot\vec{E} = 0$ & $\vec{\nabla}\cdot\vec{B} = 0$) as $\vec{E}, \vec{B}$ are functions of 'z'

& $\vec{B} = \dfrac{\hat{k} \times \vec{E}}{c}$ where $\hat{k}$ is in the directio of propagation.

Energy for an EM Wave

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$

& following Dave's Notes

$$S = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$

with $\dfrac{dU}{dt} = -(\vec{\nabla}\cdot\vec{S}) = -\int \vec{S}\cdot d\vec{a}$

Energy Density vector.

(We shall avoid discussions on polarization, dispersion, etc for this course).

Let us have a look at a few problems.

Problem 1

$$\vec{B} = B_0 \sin(kx - \omega t)\,\hat{j}$$

a) What is the wavelength λ of the wave?

b) Write an expression for the $\vec{E}$ associated to this magnetic field.

c) What direction is it moving?

d) Find $\vec{S}$

e) The wave is totally reflected by a thin perfectly conducting sheet lying in the yz plane at $x=0$. What is the radiation pressure on the sheet?

f) Find the expression for the reflected wave.

a) $\lambda = \dfrac{2\pi}{k}$ (discuss)

b) c) Propagation is towards $\hat{i}$

e. $\vec{B} = \dfrac{\hat{i} \times \vec{E}}{c}$

$\Rightarrow \vec{E}$ must be $(-\hat{k})$ & $|\vec{E}| = c B_0 \sin(kx - \omega t)$

d) $S = \dfrac{\vec{E} \times \vec{B}}{\mu_0} = \dfrac{c B_0^2}{\mu_0} \sin^2(kx - \omega t)\,\hat{i}$

e) We know from SR, $E = |P| c$ for light waves.

$\therefore$ for $|\vec{S}|$ which has the energy density,

$$|P_{density}| = \frac{|\vec{S}|}{c}$$

Let us assume that the change in momentum occurs over time Δt.

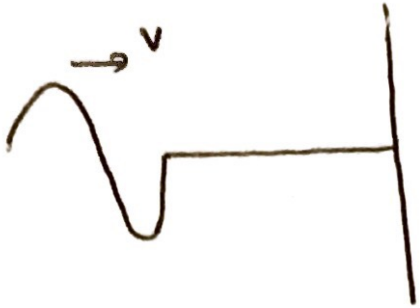
$$\Delta P_{density} = 2 \frac{|\vec{S}|}{c} \quad \text{in time } \Delta t$$

$$P_{ressure} = \frac{\Delta P_{density}}{\Delta t} = 2 \frac{|\vec{S}|}{c}$$

$$\langle Pressure \rangle = \frac{2 \langle S \rangle}{c}$$

$$= \frac{B_0^2}{\mu_0}$$

i)



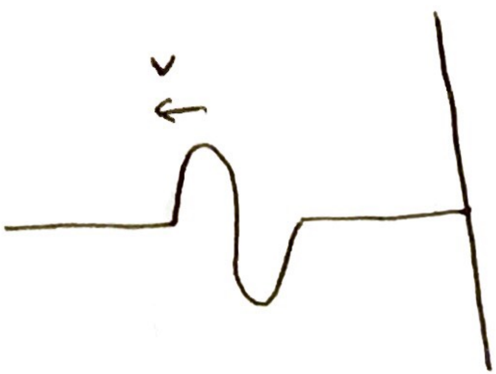
x = 0

$\vec{E} = c B_0 \sin(kx - \omega t)(-\hat{k})$

phantom wave

$\therefore$ Phantom wave

$= c B_0 \sin(kx + \omega t)(-\hat{k})$



Resut

$\therefore$ Reflected wave

$= c B_0 \sin(kx + \omega t)(-\hat{k})$

Total wave $= c B_0 \left(\sin(kx - \omega t) + \sin(kx + \omega t) \right)(-\hat{k})$

Now, lets clarify a few lies,

Battery spends energy and energy is dissipated by the wire.

Here, we are making an overall estimate of where energy is spent by the system and how its dissipated.

Let us instead look at the mechanism now, Consider ~~a~~ wire or resistor element, it must have energy conserved. This is because the electrons in it continue to move with same drift speed under the influence of the battery.

<u>Problem 2</u>

Calculate the poynting vector associated to the wire and the energy change in the wire. You may consider $\vec{E}$ right outside the wire as equal to $\vec{E}$. The wire is ohmic. (thickness d, area A)

$$\vec{B} = \frac{\mu_0 I}{2\pi d} \hat{\theta}$$

$$\vec{E} = E\hat{z}$$

$$\therefore S = \frac{\vec{E}\times\vec{B}}{\mu_0} = -\frac{\mu_0 I E}{2\pi d \mu_0}\hat{r} = \frac{-IE}{2\pi d}\hat{r} = -\frac{JI}{2\pi d}\hat{r}$$

$$\therefore \text{Power input} = \frac{dU}{dt} = -\int \vec{S}\cdot\vec{da}$$

$$= \frac{JI}{\sigma 2\pi d} 2\pi d L$$

$$= I^2 R$$

This input power increases the ~~heat~~ temperature of the resistor which is then lost as heat during equilibrium.

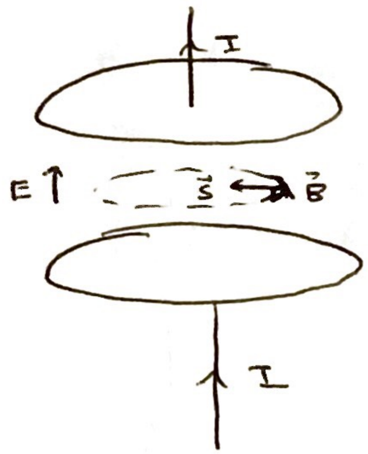
But that wasn't the only thing we obsured.

Consider a capacitor, we told you that the field has a certain energy when it' has been set up but never described the process.

So here we go,

Problem 3

A capacitor with circular plates of radius R is charged by a constant current I. As $\vec{E}$ increases between the plates, energy density increases. ∴ energy must be flowing into the capacitor. Calculate the poynting vector at radius 'r' for the setup. Verify that it explains the increase in energy of the capacitor.



For $S \neq 0$, B must be $\neq 0$

∴ Using corrected ampere's law,

$$\oint \vec{B} \cdot dl = \int \mu_0 \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a}$$

$$B = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \times \frac{\pi r^2}{2\pi r}$$

$$= \frac{\epsilon_0 \mu_0 r}{2} \frac{\partial \vec{E}}{\partial t}$$

$$S = \left| \frac{\vec{E} \times \vec{B}}{\mu_0} \right| \quad \text{points inwards}$$

$$= \frac{\epsilon_0 \mu_0 r}{2} \frac{\dot{E}}{\mu_0} \frac{\partial \dot{E}}{\partial t} = \frac{\epsilon_0 r}{2} E \frac{\partial E}{\partial t}$$

$$P = -\int \vec{S} \cdot d\vec{a}$$

$$= \frac{\epsilon_0 r}{2} E \frac{\partial E}{\partial t} 2\pi r h$$

$$= \frac{d}{dt} \underbrace{\left(\pi r^2 h \times \frac{\epsilon_0 E^2}{2} \right)}_{U \text{ inside the radius } r.}$$