Lecture 12

We had earlier studied that using Ampere's law,

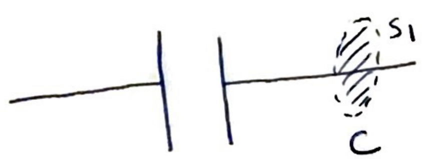
$$\oint \vec{B}\cdot\vec{dl} = \mu_0 I_{enc}$$

Rather,

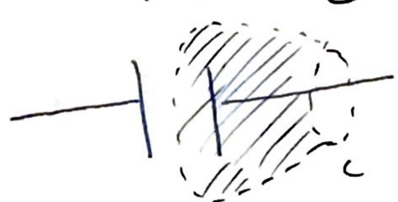
$$\oint_C \vec{B}\cdot\vec{dl} = \mu_0 \int_S \vec{J}\cdot d\vec{A}$$

where S is a surface enclosed by C. From our earlier discussions, we had noted that this S can be any surface bounded by C.

Let us consider the following example,



$$\int \vec{J}\cdot d\vec{A} = I$$



$$\int \vec{J}\cdot d\vec{A} = 0$$

(no current crosses the surface)

$\oint \vec{B}\cdot\vec{dl} = \mu_0 \int \vec{J}\cdot d\vec{A}$ is clearly incorrect. Consequently, the differential version must be too. i.e $\nabla \times B = \mu_0 J$ is incorrect.

Further

$\vec{\nabla}\times\vec{B} = \mu_0 J$ holds only for steady currents.

$$\vec{\nabla}\cdot(\vec{\nabla}\times B) = 0 = \mu_0(\vec{\nabla}\cdot\vec{J})$$

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} \text{ (from continuity equation)}$$

$$\vec{\nabla}\cdot\vec{J} \neq 0 \text{ for changing currents}$$

To make

$\vec{\nabla}.(\vec{\nabla} \times \vec{B}) = 0$ (I just need to add a term that has a opposite contribution to $\vec{\nabla}.\vec{J} = 0$)

$$\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\underbrace{(\varepsilon_0 \vec{\nabla}.\vec{E})}_{\text{from Gauss Law}} = -\vec{\nabla}.\varepsilon_0\left(\frac{\partial E}{\partial t}\right)$$

$$\therefore \vec{\nabla} \times \vec{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$ solves our problem.

I have a slightly off beat way of looking at this equation. You can decide for yourself if the picture helps.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ (electric field created by changing magnetic dipole field)

Similarly

$$\vec{\nabla} \times \vec{B} = \underbrace{(\bullet)}_{\text{proportionality factor}}\frac{\partial \vec{E}}{\partial t}$$ (due to charge dipoles)

As charges can exist as monopoles,

$$\vec{\nabla} \times \vec{B} = (\ )\vec{J} + (\ )\frac{\partial E}{\partial t}$$

If magnetic monopoles existed,

$$\vec{\nabla} \times \vec{E} = (\ )\underbrace{\vec{\beta}}_{\text{current of magnetic mopoles}} + (\ )\frac{\partial \vec{B}}{\partial t}$$

Coming back to our initial discussion, the second term appears due to changing current or accumulation of charges.

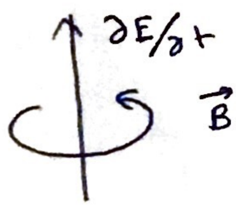
$$\therefore \quad J_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Displacement Current Density

(Historical term - holds little importance now)

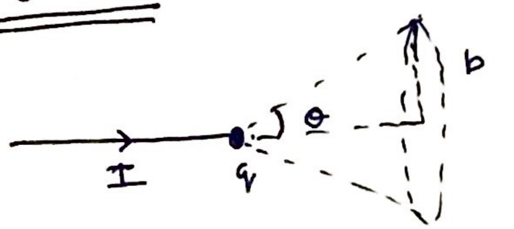
We instead interpret it as changing ~~current~~ electric field leading to magnetic field.

(Recollect the similar scenario for Faraday's electric field due to changing magnetic field).



Due to the very small value of ε_0, we shall ignore this term for quasi static cases or slowly varying electric fields.

Problem 1



A half-infinite wire carries current I from negative infinity to origin where it builds up as a point charge q. Consider the circle of radius 'b' and subtends an angle '2θ' with respect to the charge.

Find $\oint \vec{B} \cdot \vec{dl}$ for this curve.

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \int \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \cdot \vec{da}$$

$[\vec{J} = 0$ through the surface$]$

$$= \mu_0 \varepsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot \vec{da}$$

$$= \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(\int \vec{E} \cdot \vec{da}\right)$$

$[$ Solid angle $= 2\pi [1 - \cos\theta]]$

$$= \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left[\frac{q\, 2\pi [1-\cos\theta]}{\varepsilon_0 \times 4\pi}\right]$$

$$= \frac{\mu_0 \varepsilon_0}{\varepsilon_0} \frac{1}{2} [1-\cos\theta]\, I$$

$$= \frac{\mu_0}{2} [1-\cos\theta]\, I$$

Problem 2

A solenoid of radius 'R' & 'n' turns per unit length has current $I = I_0 \cos(\omega t)$, flowing through it.

1) Calculate B due to the current.

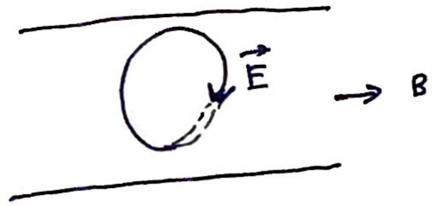
2) Calculate E due to changing $\vec{B}$.

3). Changing $\vec{E}$ causes $\vec{B}$. Calculate ΔB^{ind} due to $\vec{E}$, where $\Delta B^{ind} = B^{ind}(r,t) - B(0,t)$.

4) Show that $\Delta B/B$ is very small for frequencies encountered in daily life.

1) $\vec{B} = \mu_0 n I_0 \cos(\omega t)$

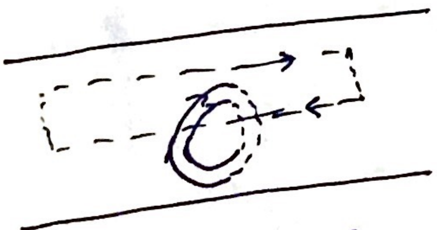
2) $\int \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{a}$



$E \times 2\pi r = +\mu_0 n I \omega \sin(\omega t)\, \pi r^2$

$\vec{E} = +\dfrac{\mu_0 n I \omega r \sin(\omega t)}{2}$

3)



(I decide the direction already by other sl...)

$$\int \vec{B} \cdot d\vec{s} = \mu_0 \int \left(J + \varepsilon_0 \frac{\partial E}{\partial t}\right) \cdot d\vec{a}$$

$$= \mu_0 \int \vec{J} \cdot d\vec{a} + \mu_0 \varepsilon_0 \frac{\partial \phi_E}{\partial t}$$

$$l\left(B(0,t) - B(r,t)\right) = \mu_0 \varepsilon_0 \frac{\partial \phi_E}{\partial t}$$

$$-\cancel{l}\, \Delta(B(r,t)) = \mu_0 \varepsilon_0 \frac{\mu_0 n I \omega^2 r^2}{4} \cos(\omega t)\, \cancel{l}$$

$$\Rightarrow \Delta B(r,t) = \frac{\mu_0^2 \varepsilon_0 n I_0 \omega^2 r^2}{4} \cos(\omega t)$$

Problem 3

If Magnetic monopoles existed, Maxwell's equations would look like

$$\vec{\nabla}\cdot\vec{E} = \rho_e/\varepsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = \mu_0\rho_m$$

$$\vec{\nabla}\times\vec{E} = -\mu_0\vec{J}_m - \frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}_e + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

Assume a particle with q_m and no q_E. It passes through a loop wire with self Inductance L.

a> Write down integral form of Faraday's law from these equations

b> Using self Inductance, find $\frac{\partial I(t)}{\partial t}$ in the loop.

c> Integrate over time and rewrite as total change in variables.

d> Assuming that there is no initial current in loop, what current is flowing after the monopole has passed through? (Assume it started arbitrarily far and is again arbitrarily far away).

a>

$$\int(\vec{\nabla}\times\vec{E})\cdot d\vec{A} = -\mu_0\int\vec{J}_m\cdot d\vec{a} - \int\frac{\partial B}{\partial t}\cdot da$$

$$\therefore \oint\vec{E}\cdot d\vec{l} = -\mu_0 I^m_{enc} - \frac{\partial\Phi_B}{\partial t}$$

b>

$$\mathcal{E} = -L\frac{dI_e}{dt}$$

$$L\frac{dI_e}{dt} = \mu_0 I^m_{enc} + \frac{\partial\Phi_B}{\partial t}$$

c>

$$L\frac{dI_e}{dt} = \mu_0\frac{\partial Q_{m\,enc}(t)}{\partial t} + \frac{\partial\Phi_B(t)}{\partial t}$$

$$L\,\Delta I = \mu_0\,\Delta Q_m + \Delta\Phi_B \Rightarrow \Delta I = \frac{\mu_0}{L}\Delta Q_m + \frac{1}{L}\Delta\Phi_B$$

d)

$$\Delta \Phi_B = 0$$

$$\Delta I = \frac{\mu_0 \Delta Q_m}{L}$$

$$\therefore I_f = \frac{\mu_0 q_m}{L}$$