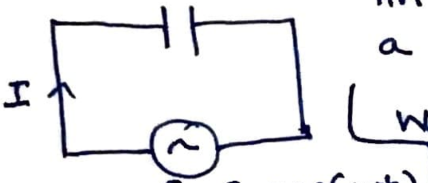
Lecture 10

We shall take a slightly different approach that what was followed in the class. We will discuss a few problem solving methods and you can choose what you prefer.
(quick comment on Transient & steady state solution)

C- Impedance



$\mathcal{E} = \mathcal{E}_0 \cos(\omega t)$

An oscillating $\mathcal{E}$ can be generated by varying a magnetic field through a wire loop.

We can solve for the circuit assuming $I = I_0 \cos(\omega t + \phi)$ but it is far more convenient to use $\tilde{I} = I_0 e^{i(\omega t + \phi)}$.

This makes, $I = \mathrm{Re}(\tilde{I})$ & $V = \mathrm{Re}(\tilde{V})$.

Using loop law,

$$\tilde{\mathcal{E}} - \tilde{Q}/C = 0$$

$$\Rightarrow \tilde{Q} = C \mathcal{E}_0 e^{i\omega t}$$

$$\Rightarrow \tilde{I} = C\mathcal{E}_0 i\omega e^{i\omega t}$$

$$= \frac{\mathcal{E}_0}{(1/i\omega C)} e^{i\omega t}$$

There are two ways of looking at this

$$Z_C = 1/i\omega C$$

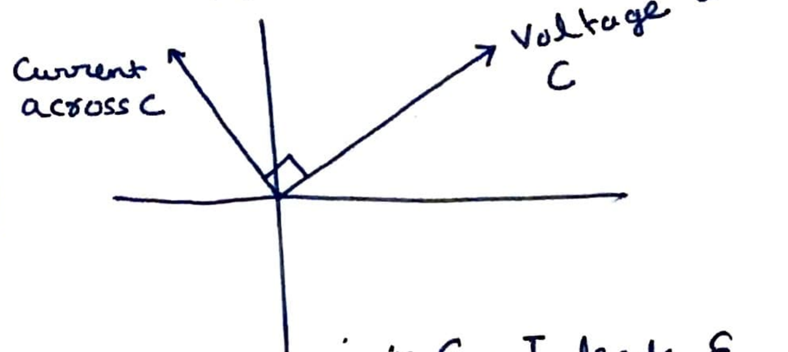
↑ includes phase factor

$$\tilde{I} = \tilde{\mathcal{E}}/Z_C$$

Reactance $= 1/\omega C$

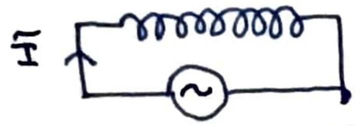
$$\tilde{I} = \frac{\mathcal{E}_0}{(1/\omega C)} i e^{i\omega t}$$

$$= \frac{\mathcal{E}_0}{1/\omega C} e^{i(\omega t + \pi/2)}$$



$\therefore$ in C, I leads $\mathcal{E}$

L - Impedance



$\mathcal{E} = \mathcal{E}_0 \cos(\omega t)$

Using loop law,

$$\tilde{\mathcal{E}} - L\frac{d\tilde{I}}{dt} = 0$$

$$\Rightarrow \tilde{\mathcal{E}} = L\frac{d\tilde{I}}{dt} \quad \left(\text{assuming } \tilde{I} = I_0 e^{i(\omega t + \phi)}\right)$$

$$\Rightarrow \tilde{\mathcal{E}} = L I_0 i\omega \, e^{i(\omega t + \phi)}$$

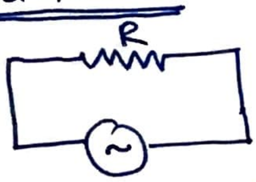
$$\Rightarrow \tilde{I} = \tilde{\mathcal{E}}/i\omega L$$

$\therefore Z_L = i\omega L$

& Reactiance of $L = \omega L$ with phase factor of $-\pi/2$.

$$\therefore I = \mathcal{E}_0/\omega L \; e^{i(\omega t - \pi/2)}$$

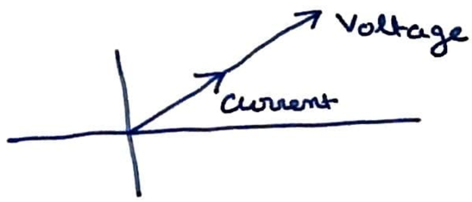
Resistance



$\mathcal{E} = \mathcal{E}_0 \cos(\omega t)$

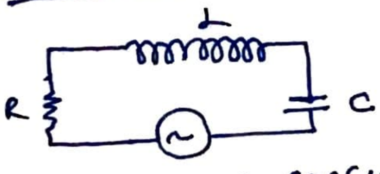
$$\tilde{\mathcal{E}} - \tilde{I}R = 0$$

$$\Rightarrow \tilde{I} = \tilde{\mathcal{E}}/R$$



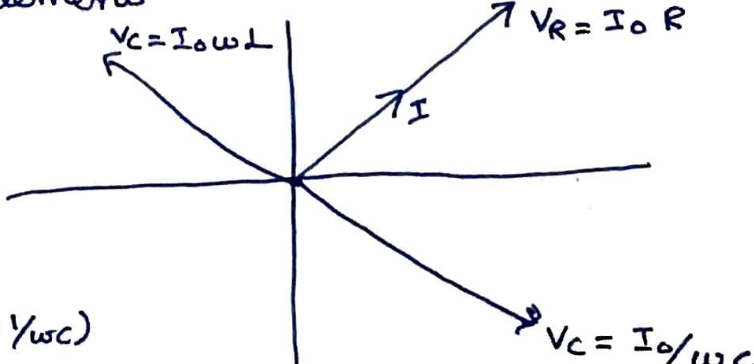
Mnemonic :- ELI the ICE man

Series RLC Circuit

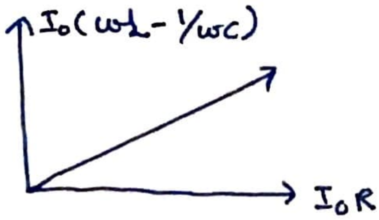


$\mathcal{E} = \mathcal{E}_0 \cos(\omega t)$

The same current flows through the circuit elements



$\therefore$ Net Voltage



$$= I_0 \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

& $\tan\phi = \dfrac{V_L - V_C}{V_R}$

$$\phi = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right)$$

$\llcorner$ phase by which voltage leads current.

Alternately

In series $\quad Z_{net} = Z_R + Z_L + Z_C$

$$\tilde{I} = \frac{\mathcal{E}}{Z_{net}} = \frac{\mathcal{E}_0 e^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}}$$

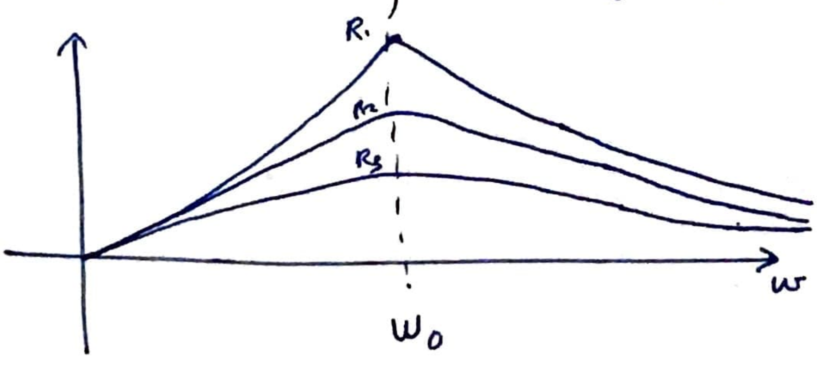
$$= \frac{\mathcal{E}_0 e^{i\omega t}}{\left(R + i\left(\omega L - \frac{1}{\omega C}\right)\right)}$$

$$= \frac{\mathcal{E}_0 e^{i\omega t}\left(R - i\left(\omega L - \frac{1}{\omega C}\right)\right)}{\left(R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right)}$$

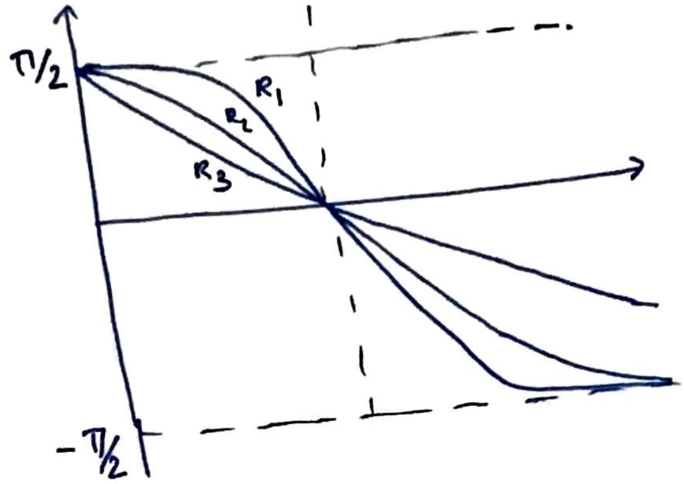
$$= \frac{\mathcal{E}_0 e^{i\omega t}\left(R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right)^{1/2} e^{i\phi}}{\left(R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right)}$$

where $\phi = \tan^{-1}\left(\dfrac{-(\omega L - 1/\omega C)}{R}\right)$

$$= \frac{\mathcal{E}_0 e^{i(\omega t + \phi)}}{\left(R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right)^{1/2}}$$



$R_1 < R_2 < R_3$



Further $Q = \omega L / R$ (no of radians in which energy falls to 1/e of undriven circuit)

Width of $I_0(\omega)$ curve

An useful quantity is the value of current for which power is half of maximum.

$$I_0 = I_{0max}/\sqrt{2}$$

This is possible for $\left(\omega L - \frac{1}{\omega C}\right) = R$

Now, $$\omega L - \frac{1}{\omega C} = \omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0}\right) - \frac{1}{\omega_0 C\left(1 + \frac{\Delta\omega}{\omega_0}\right)}$$

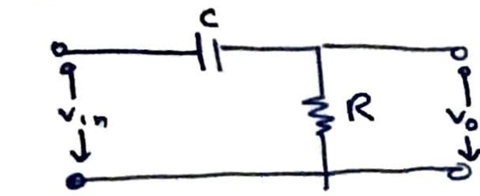
$$= \omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0} - \frac{1}{1 + \frac{\Delta\omega}{\omega_0}}\right)$$

$$= \omega_0 L\left(\frac{2\Delta\omega}{\omega_0}\right)$$

$$\therefore \quad \omega_0 L\left(\frac{2\Delta\omega}{\omega_0}\right) = R$$

$$\Rightarrow \quad 2\Delta\omega = \frac{\omega_0}{Q}$$

Problem 1



$V_{in} = V_0 \sin(\omega t)$

a) Find total impedance

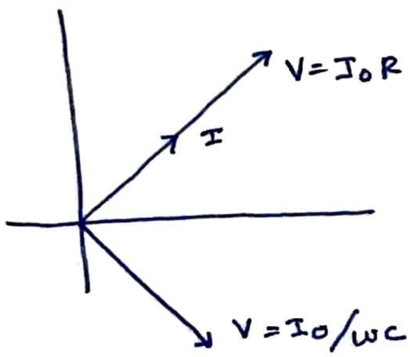
b) Find amplitude & phase of current

c) Find amplitude & phase of output voltage

d) Find $|V_{out,0}| / |V_{in,0}|$

e) Discuss the nature of this circuit.

a)



$$\therefore Z = R + \frac{1}{i\omega C}$$

$$= R - \frac{i}{\omega C}$$

$$= \sqrt{R^2 + (1/\omega C)^2}\, e^{i\tan^{-1}\left(-\frac{1}{\omega C R}\right)}$$

b)

$$\tilde{I} = \frac{V_0\, e^{i(\omega t - \pi/2)}}{\sqrt{R^2 + (1/\omega C)^2}\, e^{i\tan^{-1}(-1/\omega C R)}}$$

$$I_0 = \frac{V_0}{\sqrt{R^2 + (1/\omega C)}} \quad \& \quad \phi = \tan^{-1}\left(-1/\omega C R\right)$$

c)

$$V_{out} = \tilde{I} R$$

$$= \frac{V_0 R}{\sqrt{R^2 + (1/\omega C)^2}}\, e^{i(\omega t - \pi/2 + \tan^{-1}(1/\omega C R))}$$

$\therefore$ Amplitude $= \dfrac{V_0 R}{\sqrt{R^2 + (1/\omega C)^2}}$ Phase $\phi' = \tan^{-1}(-1/\omega C R)$

d)

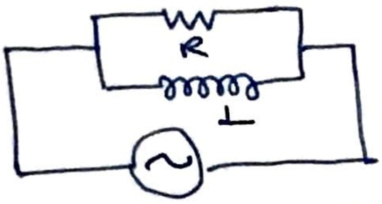
$$\left| \frac{V_{out,0}}{V_{in,0}} \right| = \frac{V_0 R}{V_0 Z} = R/Z$$

e) For small ω, $V_{out} \to 0$
large ω, $V_{out} \to V_0/R$

$\therefore$ only signal with high frequency is measured.

Problem 2

i) Find the current response & phase for



$\mathcal{E} = \mathcal{E}_0 \cos \omega t$

$$Z = \left(\frac{1}{R} + \frac{1}{i\omega L}\right)^{-1}$$

$$= \left(\frac{\omega L - iR}{\omega L R}\right)^{-1}$$

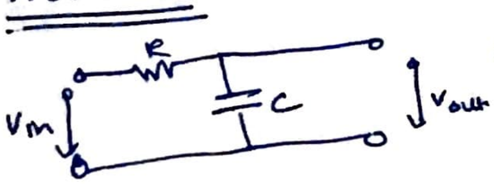
$$= \frac{\omega L R}{\omega L - iR}$$

$$= \frac{\omega L R}{((\omega L)^2 + R^2)^{1/2}} e^{i\tan^{-1}(-R/\omega L)}$$

$$\therefore I = \frac{\mathcal{E}_0 e^{i\omega t}}{Z}$$

$$= \mathcal{E}_0 \left(1/R^2 + \frac{1}{\omega L^2}\right)^{1/2} e^{i(\omega t - \tan^{-1}(R/\omega L))}$$

Problem 3



Characterize this circuit.

$$Z = R + \frac{1}{i\omega C}$$

$$= R - i/\omega C$$

$$= \sqrt{R^2 + (1/\omega C)^2}\; e^{-i\tan^{-1}\left(-\frac{1}{\omega CR}\right)}$$

$$\tilde{I} = V_0 e^{i\omega t}/Z = \frac{V_0}{\sqrt{R^2 + (1/\omega C)^2}} e^{i(\omega t + \tan^{-1}(1/\omega CR))}$$

$$\tilde{V}_{out} = \tilde{I} Z_c = \frac{V_0\, e^{i(\omega t - \pi/2 - \tan^{-1}(-1/\omega CR))}}{\sqrt{R^2 + (1/\omega C)^2}\;\omega C}$$

$$= \frac{V_0}{\sqrt{1 + (\omega CR)^2}} e^{i(\omega t - \pi/2 + \tan^{-1}(1/\omega CR))}$$

V_{out} is significant for $\omega CR \ll 1$. $V_{out} \to 0$ for $\omega CR \gg 1$.

$\therefore$ it is a low pass filter

Problem 2 – Back EMF in Inductors

Consider the simple *RL* circuit shown in Figure 1. The purpose of the resistance *R* is to limit the current drawn from the AC source. Suppose you turn on the switch, which we assume is an ideal switch that can turn on and off instantaneously. The current starts flowing through the circuit and the back EMF produced by the inductor results in a slow growth of the current, $dI / dt = \mathcal{E}_0 / L$. Once the current reaches steady state ($dI / dt = 0$) you turn off the switch. We will approximate the "OFF" position by assuming a very high resistance value R_a across the terminals of the open switch, where the subscript *a* stands for air. Note that $R_a \gg R$. The sudden loss of current flow will result in a back EMF generated by the inductor.

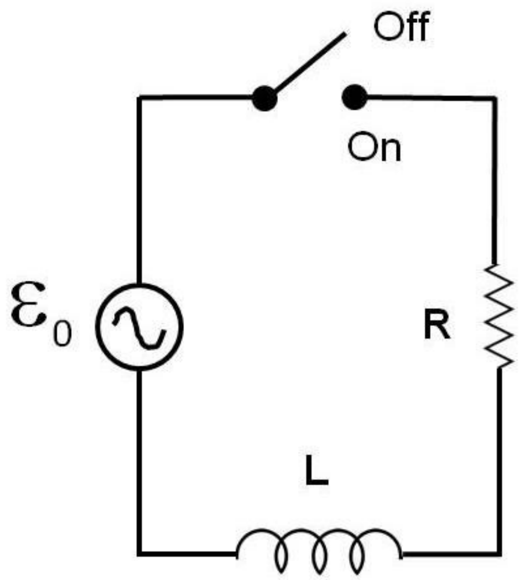


Figure 1: *RL* circuit with a switch.

(a) Show that the current through the inductor dies down as $I(t) \approx I_0 e^{-tR_a/L}$, where I_0 is the current in the switch "ON" position. [1pt]

(b) For the purpose of part (b) only, adopt the following specific values: $\mathcal{E}_0 = 110$ V, frequency $\omega/2\pi = 60$ Hz, $R = 100\ \Omega$, $L = 1$ mH and $R_a = 10^6\ \Omega$. Assume that the inductor can stand up to 10 kV of voltage across its input terminals. Compute the peak value of the back EMF across the inductor. You should find that the inductor burns up[1]. [1pt]

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[1] This experiment is not recommended!

(c) Having been badly burnt by your first experiment, you come to appreciate the importance of reactance in circuits. Clearly, you need to cancel the reactance of the inductor so that huge back EMF is not generated. Relative to inductors, capacitors have a negative reactance. Keeping this in mind you devise the clever circuit shown in Figure 2 to limit the back EMF. In this circuit, R and L are as before[2] but you will have a choice in the value of C. Find the value of C, expressed in terms of R and L, that will result in zero reactance across terminals A and B. [1pt]

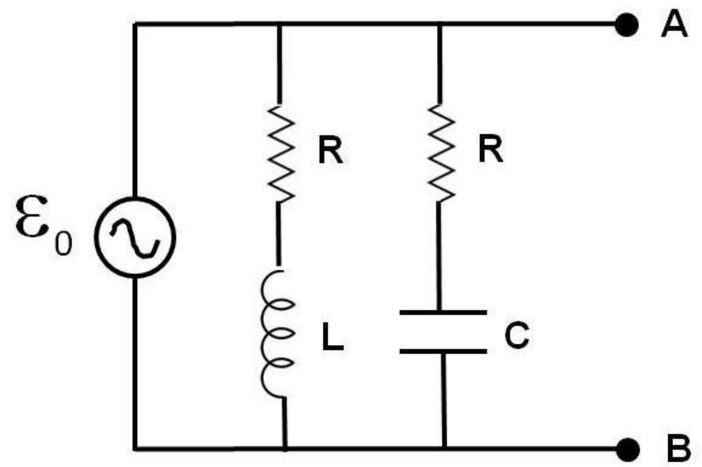

Figure 2

Use the value of C you found in part (c) in the following two parts.

(d) Your goal is now to compute the time-averaged power dissipation in the circuit. In the first method, compute Z_R, the net resistive load across the terminals of the EMF source. Then find the average dissipated power knowing $\bar{P} = (1/2)\mathcal{E}_0^2/Z_R$. [1pt]

(e) In the second method, compute the power dissipated across each resistive element and sum up to yield the total power dissipation. Compare the power values derived by the two methods. [Warning: this can be solved efficiently and rapidly with phasor algebra, provided you are aware of the nuances of phasor multiplication. The alternative approach is to use real functions but with care also]. [1pt]

[2] In parts (c), (d) and (e) we do NOT ascribe specific values to R, $\mathcal{E}_0$ and L like we did in part (b).

Problem 2 - Back EMF in Inductors

a [1pt]

As soon as the switch is turned on current starts to flow and application of Kirckoff's law yields

$$-L\frac{dI}{dt} + RI = \mathcal{E}_0$$

. Note that inductor opposed the action of turning on and produces and EMF to counteract the EMF of the supply. $I(0) = 0$ and so at early times we can ignore the RI term and find

$$\frac{dI}{dt} = \frac{\mathcal{E}_0}{L}$$

In steady state, the amplitude of the current flowing through this L-R circuit is easily computed (L and R are in series) and found to be

$$I = \frac{\mathcal{E}_0}{\sqrt{R^2 + L^2\omega^2}}$$

For the specific value of $R = 100$ ohm, $\mathcal{E}_0 = 110$ V and $L = 1$ mH we find $I = 1.1$ A (dominated by the resistor).
Once the switch is turned off the resistance in the circuit is $R+R_a$ which is entirely dominate by R_a. The rms voltage drop across this resistor is IR_a (we can ignore the drop across R and the voltage provided by the source) which is approximately 1.1×10^6 V. This will surely blow up our inductor. Ignoring this physical discussion we have mathematically

$$L\frac{dI}{dt} + IR_a = 0$$

The boundary condition is $I(0) = I$ (with time zero being the time at which the switch was turned off). The rms current through the sicruit decays as $I\exp(-\frac{t}{(L/R)})$. The time constant is small, 1 ns (whence, anticipating this situation, we invoked an ideal switch!).

b [1pt]

The impedance from the resistance and inductor is

$$Z_1 = R + iL\omega$$

whereas that from the resistance and capacitor is

$$Z_2 = R + \frac{1}{iC\omega}$$

The admittance across the terminal A and B is

$$\begin{aligned} Y &= \frac{1}{Z_1} + \frac{1}{Z_2} \\ &= \frac{1}{R + iL} + \frac{1}{R + \frac{1}{iC\omega}} \\ &= \frac{2R + iA}{B + iRA} \end{aligned}$$

where $A = L\omega - \frac{1}{(C\omega)}$ and $B = R^2 2 + L/C$. The impedance is

$$\begin{aligned} Z &= \frac{B + iRA}{2R + iA} \\ &= \frac{B + iRA}{2R + iA} \times \frac{2R - iA}{2r - iA} \\ &= \frac{2BR + RA^2 + iA(2R^2 - B)}{4R^2 + A^2} \\ &= Z_R + iZ_Y \end{aligned}$$

The reactance can be eliminated (i.e. set to zero) by choosing C so that $2R^2 = B$ leading the requirement

$$C = \frac{L}{R^2}$$

For this choice of C the resistance is

$$Z_R = \frac{R(4R^2 + A^2}{4R^2 + A^2} = R$$

c [1pt]

Thetotal current flowing out of the EMF source is

$$\mathcal{I} = \frac{\mathcal{E}_0}{Z} = \frac{\mathcal{E}_0}{R}$$

Note that the impedance is entirely resistive and thus the current is in phase with the EMF. The mean power is trivially found to be

$$\bar{P} = 1/2\mathcal{E}_0 I_0 = \frac{\mathcal{E}_0^2}{R} \tag{1}$$

d [1pt]

Let I_1 and I_2 be the current through impedance Z_1 and Z_2, respectively.

$$I_1 = \frac{\mathcal{E}_0}{Z_1} = \frac{\mathcal{E}_0}{R + i\omega L} = \frac{\mathcal{E}_0}{\sqrt{R^2 + L^2\omega^2}} = I_1 exp(i\phi_1)$$

where $\tan\phi_1 = -L\omega/R$ and $I_1 = \mathcal{E}_0/\sqrt{R^2 = L^2\omega^2}$
We now compute I_2 and use the above equation to eliminate C:

$$I_2 = \frac{\mathcal{E}_0}{Z_2} = \frac{\mathcal{E}_0}{R + \frac{1}{iC\omega}} = \frac{\mathcal{E}_0(L\omega/R)}{\sqrt{R^2 + L^2\omega^2}} exp(i\phi_2)$$

where $\tan\phi_2 = R/(L\omega)$
We now compute the dissipation across each of the two resistors using real functions and later with phases. The voltage drop across the resistor has the same phase as the current. Thus both current and voltage are the same form., $A\cos(\omega t + \phi)$ where A is the amplitude (say, A for current and $A_V = RA$ for the voltage) and ϕ is the phase. The time aevraged dissipated power is $(1/2)A^2R$. The sum of the power dissipated across the two resistors is thus

$$\bar{P} = \frac{1}{2}(I_1^2 R + I_2^R) = \frac{1}{2}\frac{\mathcal{E}_0^2}{R} \tag{2}$$

which matches that found previously (equation 1).
We will now compute the dissipated power using phasor algebra. There is a subtle point here and that has to do with the phase algebra. The power dissipated across any element(resistor or otherwise) is not the product of the phasor (say, current and voltage) will result in a signal of the form $\exp(i2\omega t + \phi_I + \phi_V)$ where ϕ_I is the phase of the current and ϕ_V is the phases of the voltage. This is the most certainly not the work that has been across the element. The right expression is VI^*. For a resistance, $V + RI$ and thus the power dissipates is RII^*.
the time-averaged power dissipated in each resistor is

$$\bar{P}_1 = \frac{1}{2} I_1 I_2^* R = \frac{\mathcal{E}_0^2 R}{(R^2 + L^2\omega^2}$$

$$\bar{P}_2 = \frac{1}{2} I_2 I_2^* R = \frac{\mathcal{E}_0^2 (L\omega/R)^2 R}{(R^2 + L^2\omega^2)}$$

the sum of which is the same as that found in equation 2.