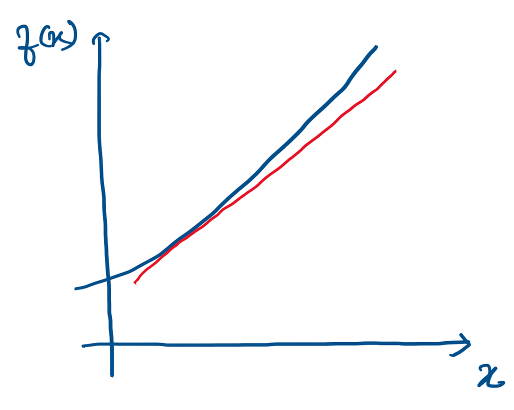
Lecture 8

Sunday, 2 February 2020 8:11 PM

Today we will avoid problems and concentrate on reviewing a few results in multivariate calculus. We will focus on problems again from next class.

Let us first start with a single variable calculus,



$\frac{df(x)}{dx}$ = slope at every point x

If we wanted to find the change of $f(x)$ over a small change dx

$$df(x) = \frac{df(x)}{dx}\, dx$$

Generalizing this over more variables,

$$dT = \underbrace{\left(\frac{\partial T}{\partial x}\right)}_{\text{change in } T \text{ due to } x.} dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$$dT = \underbrace{\left\{\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right\}}_{\text{direction of maximum increase of } T.} \cdot \underbrace{(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})}_{\text{for}}$$

$l = x\hat{x} + y\hat{y} + z\hat{z}$

$dl = dx\hat{x} + dy\hat{y} + dz\hat{z}$

Now, $\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$

$$\underbrace{\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)}_{\vec{\nabla}} T \leftarrow \text{scalar field}$$

This is not a true vector

$\vec{\nabla} T$ helps us study changes in a scalar field with respect to coordinates

What else can we do with $\vec{\nabla}$

$\vec{\nabla} T \to$ gradient

$\vec{\nabla} \cdot \vec{v} \to$ divergence

vector field.

$\vec{\nabla} \times \vec{v} \to$ curl

$\therefore \vec{\nabla} \cdot \vec{v} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$

& $\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$

A few identities

i) $\vec{\nabla}(f+g) = \nabla f + \nabla g$

ii) $\vec{\nabla} \cdot (\vec{A} + \vec{B}) = \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{B}$

iii) $\vec{\nabla} \times (\vec{A} + \vec{B}) = (\vec{\nabla} \times \vec{A}) + (\vec{\nabla} \times \vec{B})$

iv) $\vec{\nabla}(fg) = f\vec{\nabla} g + g\vec{\nabla} f$

v) $\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot \vec{\nabla} f$

vi) $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla} f)$

Second derivatives

i) $\vec{\nabla} \cdot (\vec{\nabla} T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$ $\quad (\nabla^2 T)$

ii) $\vec{\nabla} \times (\vec{\nabla} T) = 0$

iii) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$

iv) $\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \nabla^2 v$

Fundamental Theorem

$$\int_a^b (\vec{\nabla} T) \cdot d\vec{l} = T(b) - T(a)$$

$\oint (\vec{\nabla} T) \cdot dl = 0$ (independent of path)

$$\int_V (\vec{\nabla} \cdot \vec{U})\, dV = \oint_S \vec{U} \cdot d\vec{a}$$ (total flow through a closed surface)

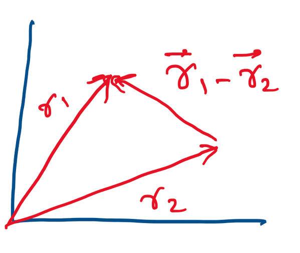
$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{l}$$

Lets start some electrostatics:

Coulomb Force

$$\vec{F}_{12} = \frac{K q_1 q_2}{r_{12}^2}\hat{r}_{12}$$

on 1 due to 2



Principle of Superposition

Interaction between any two charges are independent of any other charges.

1) What is the energy required to bring two charges from infinitely far away to r?

$$W = \int \vec{F} \cdot d\vec{s}$$

$$= \int_\infty^r -\frac{K q_1 q_2}{r^2}\, dr$$

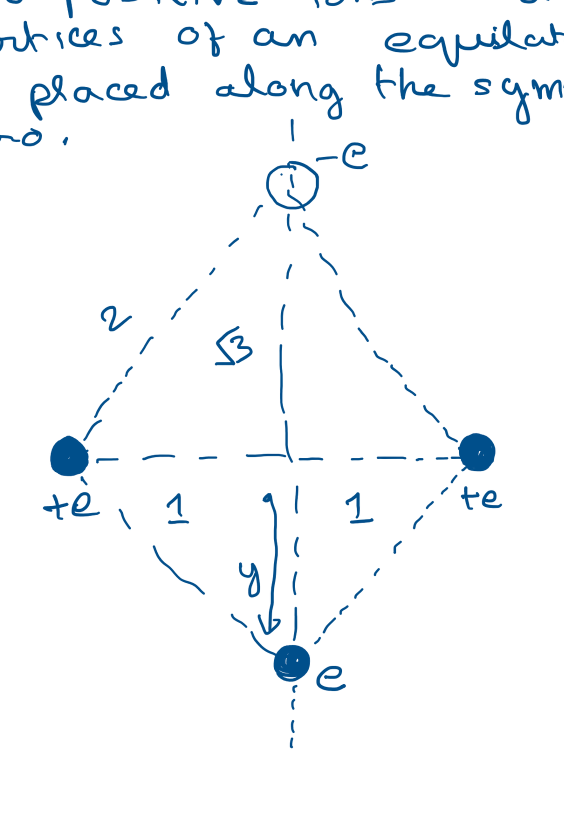
$$= \left.\frac{K q_1 q_2}{r}\right|_\infty^r$$

$$= \frac{K q_1 q_2}{r}$$

$\therefore$ Total work to assemble 3 charges

$$= \frac{K q_1 q_2}{r_{12}} + \frac{K q_2 q_3}{r_{23}} + \frac{K q_3 q_1}{r_{31}}$$

1) Two positive ions and one negative ion are fixed at the vertices of an equilateral triangle. Where can a fourth ion be placed along the symmetry axis so that force on it is zero.



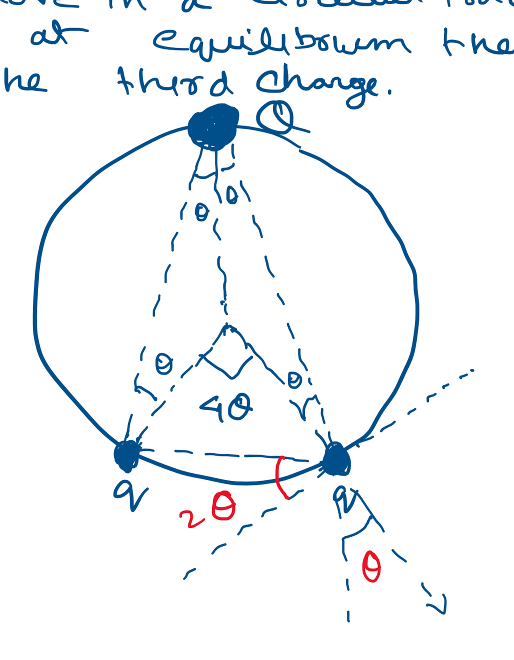
$$\frac{Ke^2}{(y+\sqrt{3})^2} = \frac{2Ke^2}{(1+y^2)}\frac{y}{(1+y^2)^{1/2}}$$

$\Rightarrow y = 0.14629$ (dont calculate)

There is a second point.

For it we get $y = 6.2$

(*) Suppose three positively charged particles are charged are constrain to move in a circular track. Suppose two of the charges are equal and at equilibrium they are 90° apart. What is the magnitude of the third charge.



The tangential force on one of the q's due to Q equals

$$\frac{KQq}{(2R\cos\theta)^2}\sin Q$$

Tangential force due to q

$$\frac{Kq^2}{(2R\sin 2\theta)^2}\cos 2\theta$$

To be in equilibrium,

$$\frac{\cancel{K}Q\cancel{q}}{(\cancel{2R}\cos\theta)^2}\sin Q = \frac{\cancel{K}q^{\cancel{2}}}{(\cancel{2R}\sin 2\theta)^2}\cos 2\theta$$

$$Q = q\frac{\cos 2\theta \cos^2\theta}{4\sin^4\theta\cos^2\theta}$$

$$= \frac{q\cos(2\theta)}{4\sin^4(\theta)}$$

Alternatively, you can define the energy and minimize it

Let us define a new quantity related to charges that can be defined independently of a second charge.

$$\vec{E} = Kq/r_{12}^2\,\hat{r}_{12}$$

electric field