Lecture 5 (Annocance)

Thursday, 23 January 2020 8:44 AM

We didnt have a lecture 4 due to MLK day. The class will therefore be more condensed
(I will also stay a discussion on Causality and instead address it through problem during Quiz Review).

Let's jump to a couple of definitions

$\Delta\tilde{x} = (c\Delta t, \Delta x, \Delta y, \Delta z)$ ← invariant interval

If we want to define a invariant velocity,

$$\tilde{V} = \frac{\Delta\tilde{x}}{\Delta\tau} = (c\,\Delta t/\Delta\tau,\ \Delta x/\Delta\tau,\ \Delta y/\Delta\tau,\ \Delta z/\Delta\tau)$$
$$= (\gamma c\,\Delta t/\Delta t,\ \gamma\,\Delta x/\Delta t,\ \gamma\,\Delta y/\Delta t,\ \gamma\,\Delta z/\Delta t)$$
$$= (\gamma c,\ \gamma v_x,\ \gamma v_y,\ \gamma v_z)$$

related to speed of object in frame

& Finally,

$$\tilde{P} = m\tilde{V} = (\gamma mc,\ \gamma m v_x,\ \gamma m v_y,\ \gamma m v_z)$$

∴ Relativistic 3 momentum = $\gamma m v$

Additionally, from Dave's solution in the class we know.

∴ Relativistic Energy $= \gamma mc^2 = KE + mc^2$

mc^2: rest mass energy.

$$\therefore \tilde{P} = (E/c,\ \vec{P})$$

Let us define one last operation.
Similar to 3 vectors in non-relativistic space, we define a scalar dot product of 4 vector that is independent of reference frames.

It is defined as,

$$\tilde{A}.\tilde{B} = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3$$

$$\therefore |\tilde{V}|^2 = \gamma^2c^2 - \gamma^2v^2 = c^2$$

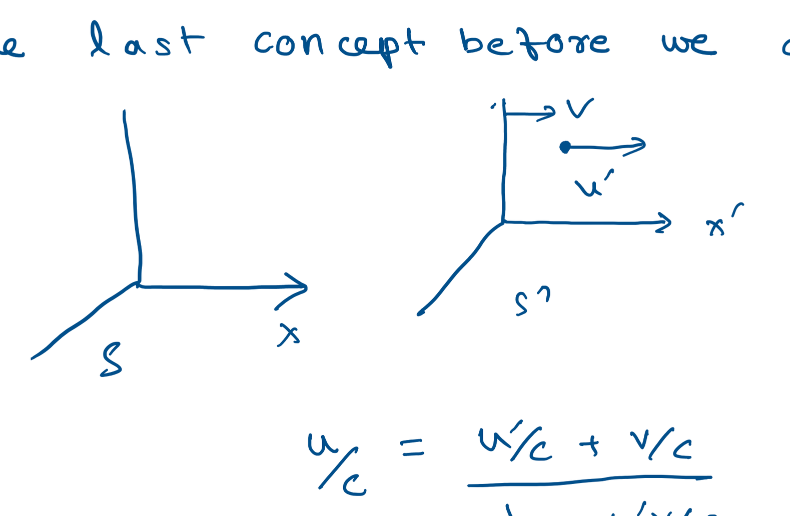
$$\therefore |\tilde{P}|^2 = m^2|\tilde{V}|^2 = m^2c^2$$

$$\therefore m^2c^2 = \frac{E^2}{c^2} - |\vec{P}|^2$$

$$\Rightarrow E^2 = m^2c^4 + |\vec{P}|^2c^2 \rightarrow \text{K.E.}$$

As laws of physics are same in all inertial frames, momentum is conserved in all frames.

One last concept before we do some problems.



$$u/c = \frac{u'/c + v/c}{1 + u'v/c^2}$$

$$\gamma_u = \frac{1}{\left(1 - \left(\frac{u'/c + v/c}{1 + u'v/c^2}\right)^2\right)^{1/2}} = \gamma_{u'}\gamma_v\left(1 + \frac{u'v}{c^2}\right)$$

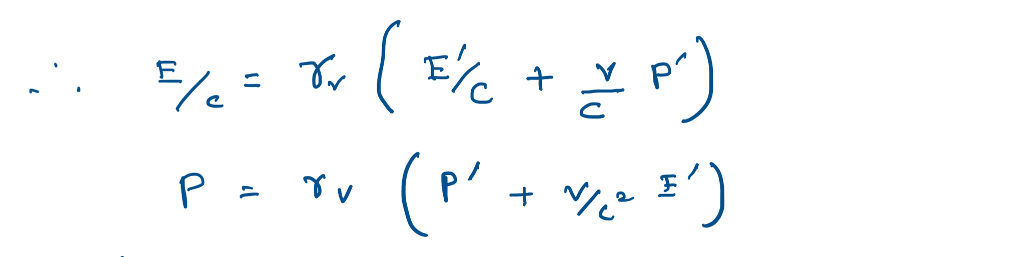
Now $E' = \gamma_{u'}mc^2$
$p' = \gamma_{u'}mu'$

Now $E = \gamma_u mc^2 = \gamma_{u'}\gamma_v\left(1 + \frac{u'v}{c^2}\right)mc^2$

$$p = \gamma_u mu = \gamma_{u'}\gamma_v\left(1 + \frac{u'v}{c^2}\right)m\,\frac{u'+v}{1+u'v/c^2}$$
$$= \gamma_{u'}\gamma_v\,m(u'+v)$$

$$\therefore E/c = \gamma_v\left(E'/c + \frac{v}{c}p'\right)$$
$$p = \gamma_v\left(p' + v/c^2\,E'\right)$$

1) A particle with mass m and energy E approaches an identical particle at rest. The collide elastically such that they both scatter at an angle θ relative to incident direction. Find θ in terms of E and m?
What is θ in the relativistic and non-relativistic limits?



$P_1 = (E/c,\ p,\ 0,\ 0)$ | $P_1' = (E'/c,\ p'\cos\theta,\ p'\sin\theta,\ 0)$

$P_2 = (mc,\ 0,\ 0,\ 0)$ | $P_2' = (E'/c,\ p'\cos\theta,\ -p'\sin\theta,\ 0)$

Conserving Energy,

$$E/c + mc = 2E'/c$$
$$\Rightarrow E' = \frac{E}{2} + \frac{mc^2}{2} \quad \text{(SR is easy)}$$
$$\& \quad p = 2p'\cos\theta$$
$$p' = p/2\cos\theta$$

$$\therefore P_1' = \left(\frac{E}{2c} + \frac{mc}{2},\ \frac{p}{2},\ \pm\frac{p}{2}\tan\theta,\ 0\right)$$

$$\Rightarrow m^2c^2 = \left(\frac{E}{2c} + \frac{mc}{2}\right)^2 - \left(\frac{p}{2}\right)^2(1 + \tan^2\theta)$$

$$\Rightarrow m^2c^2 = \left(\frac{E}{2c} + \frac{mc}{2}\right)^2 - \frac{(E^2 - m^2c^4)}{4c^2}\sec^2\theta$$

$$\Rightarrow 4m^2c^4 = (E^2 + m^2c^4 + 2Emc^2) - (E^2 - m^2c^4)\sec^2\theta$$

$$\Rightarrow \frac{E^2 - m^2c^4}{\cos^2\theta} = E^2 + 2Emc^2 - 3m^2c^4$$

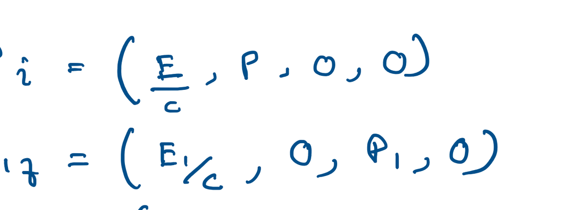
$$\Rightarrow \cos^2\theta = \frac{E^2 - m^2c^4}{E^2 + 2Emc^2 - 3m^2c^4}$$

$$\Rightarrow \cos^2\theta = \frac{E + mc^2}{E + 3mc^2}$$

In Non relativistic limit, $E \approx mc^2$

$$\cos^2\theta = 1/2$$
$$\cos\theta = 1/\sqrt{2} \quad \therefore \theta = 45^\circ$$ (we saw this earlier)

2) A particle with mass M and energy E decays into two identical particles. In the lab frame, they are emitted at angles 90° & θ. What are the energies of the created particles?



$P_i = \left(\frac{E}{c},\ P,\ 0,\ 0\right)$

$P_{1f} = (E_1/c,\ 0,\ P_1,\ 0)$

$P_{2f} = (E_2/c,\ P_2\cos\theta,\ -P_2\sin\theta,\ 0)$

Now P is conserved

$$\therefore P_i = P_{1f} + P_{2f}$$
$$\Rightarrow (P_i - P_{1f})^2 = (P_{2f})^2$$
$$\Rightarrow P_i^2 - 2P_i.P_f + P_{1f}^2 = P_{2f}^2$$
$$\Rightarrow Mc^2 - 2EE_1/c^2 + \cancel{m^2c^2} = \cancel{m^2c^2}$$

$$\therefore E_1 = M^2c^4/2E$$
$$\& \quad E_2 = E - \frac{M^2c^4}{2E}$$

4 momenta makes life easy.

3) A large mass M, moving with speed v, collides and sticks to a small mass m, initially at rest. What is the mass M_f of the resulting object? Work in the approximation when $M \gg m$.

$P_{1i} = (\gamma Mc,\ \gamma Mv,\ 0,\ 0)$ | $P_f = (E_f/c,\ P_f,\ 0,\ 0)$

$P_{2i} = (mc,\ 0,\ 0,\ 0)$

$$E_f = \gamma Mc^2 + mc^2$$
$$P_f = \gamma Mv$$

$$M_f c = \sqrt{(E_f/c)^2 - P_f^2}$$

$$M_f c = \sqrt{(\gamma Mc + mc)^2 - (\gamma Mv)^2}$$

$$M_f c = \sqrt{\gamma^2M^2c^2 + 2\gamma Mmc^2 + m^2c^2 - \gamma^2M^2v^2}$$
$$M_f c \simeq \sqrt{M^2c^2 + 2\gamma Mmc^2}$$
$$= Mc\sqrt{1 + 2\gamma\, m/M}$$
$$= Mc(1 + \gamma\, m/M)$$

4) A mass M_A decays into masses M_B and M_C. What are the energies of M_B & M_C?

$P_{Ai} = (M_Ac,\ 0,\ 0,\ 0)$ ← Always choose rest frame

$P_{Bf} = (E_B/c,\ P,\ 0,\ 0)$

$P_{Cf} = (E_C/c,\ -P,\ 0,\ 0)$

$M_Ac^2 = E_B + E_C$ | $E_B = \dfrac{c^2(M_A^2 + M_B^2 - M_C^2)}{2M_A}$

$p^2 = E_B^2/c^2 - M_B^2c^2$

$p^2 = E_C^2/c^2 - M_C^2c^2$ | $E_C = \dfrac{c^2(M_A^2 - M_B^2 + M_C^2)}{2M_A}$