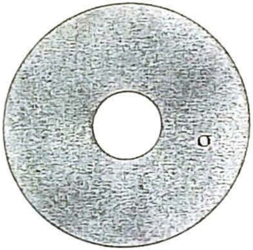
A spherical capacitor consisting of two concentric conductors centered on the origin has capacitance $C$. The region between the conductors is filled with a material with electrical conductivity $\sigma$.




At $t = 0$ the capacitor is charged with a total charge $Q_0$ on the inner conductor and $-Q_0$ on the outer conductor.

(a) **(2 points)** Write an expression for the current density $\vec{J}(r)$ in terms of $Q_0$ and $\sigma$. Use this expression to obtain an expression for the total current $I$.

(b) **(1 point)** Use the result for $I$ from part (a) and Ohm's Law to obtain an expression for the total resistance $R$ between the two conductors in terms of $\sigma$ and $C$.

(c) **(1 point)** Draw an equivalent circuit for the capacitor/resistor combination.

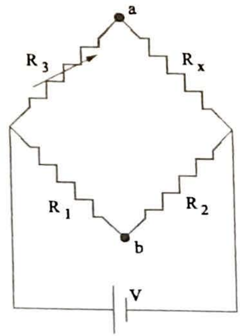
(d) **(2 points)** Write a differential equation for the charge on the capacitor.

(e) **(2 points)** Solve for the charge on the capacitor as a function of time in terms of $Q_0$ and $\sigma$.

(f) **(1 point)** What is the total energy deposited in the resistive material between the conductors, in terms of $Q_0$ and $C$ ?

(g) **(1 point)** The energy deposited per unit volume varies as a function of radius $r$ between the conductors. Derive an expression for the energy deposited per unit volume $\mathcal{E}$ in terms of $Q_0$ and the radial coordinate $r$.

A Wheatstone bridge is a device to measure the resistance of an unknown resistor $R_x$ in terms of three known resistances: $R_1$, $R_2$, and a variable resistor $R_3$. Such a device is shown in the diagram below, to which you can refer in answering the following questions.




**(a.) (2 points)** What is the voltage $V_{ab}$ between points $a$ and $b$ in terms of the applied voltage $V$ and the 4 resistances?

**(b.) (2 points)** The variable resistor $R_3$ is now adjusted so that $V_{ab} = 0$. Solve for the unknown resistance $R_x$ in terms of the 3 known values $R_1$, $R_2$, and $R_3$.

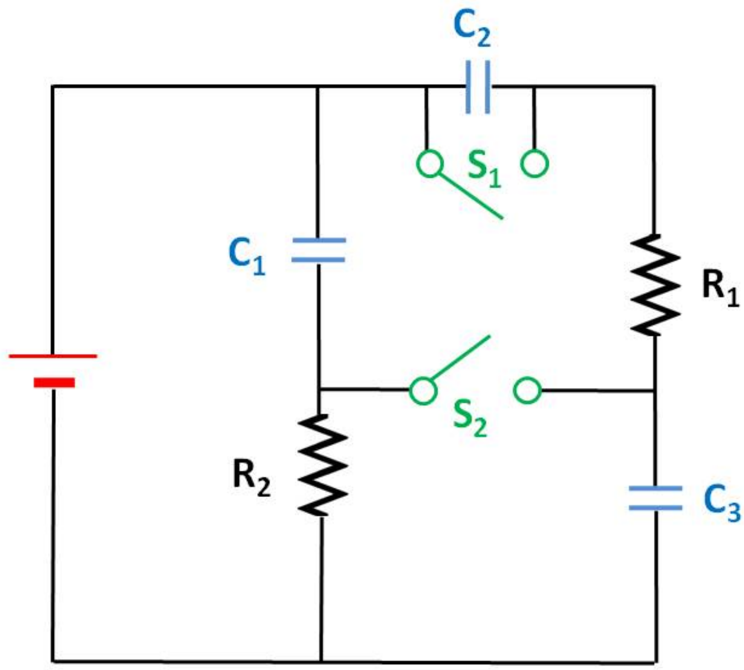
Assume that $R_1 = 10\Omega$, $R_2 = 20\Omega$, $R_3 = 30\Omega$, and $R_x = 60\Omega$. The EMF source ($V$) is disconnected and a capacitor is connected in its place. The capacitor is initially charged to a voltage of 1000 Volts. The capacitor voltage then decreases to 500 V in a time $10^{-4}$ seconds after being connected to the circuit. Please give numerical answers to parts (c), (d), and (e) below.

**(c.) (2 points)** What is the capacitance $C$ in Farads?

**(d.) (2 points)** What is the total energy in Joules that has been dissipated in the resistors during the $10^{-4}$ second time interval?

**(e.) (2 points)** Compute the fraction of the total energy that is deposited in each of the 4 resistors.

## Problem 5 – RC circuit




Consider the RC circuit above. After both switches $S_1$ and $S_2$ have been closed for a long time, the electric circuit shown above carries a steady time independent current. Take $C_1 = 2\ \mu F$, $C_2 = 6\ \mu F$, $C_3 = 4\ \mu F$, $R_1 = 2\ k\Omega$ and $R_2 = 9\ k\Omega$. The power delivered to $R_1$ is measured to be 4 W.

(a) Find the current through the circuit. [1pt]

(b) Find the charges on $C_1$, $C_2$ and $C_3$. [2pts]

(c) Find the battery voltage. [1pt]

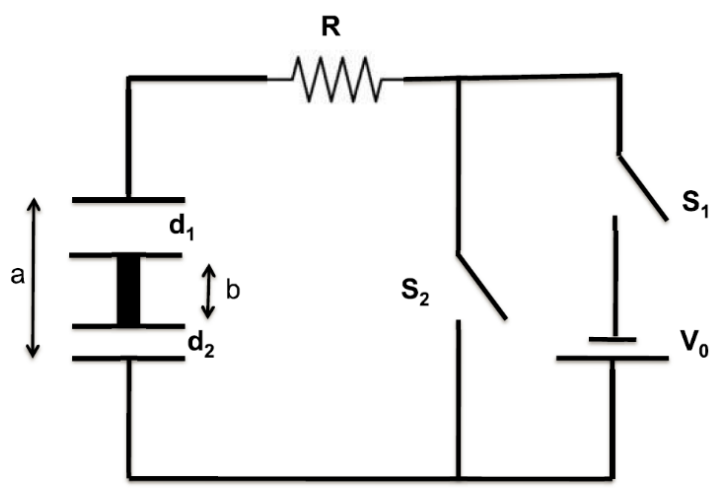
(d) The switch $S_2$ is now opened. How long does it take the current through $R_1$ to decrease to $1/e$ of its initial value? [2pts]

(e) After waiting a long time such that no currents flow through the circuit anymore the switch $S_1$ is also opened. What are the new charges across $C_1$, $C_2$ and $C_3$ after the circuit has reached its new equilibrium? [2pts]

## <u>Problem 4</u>

The figure below shows two capacitors in series, connected to a resistor $R$ and a battery. The center conductor of length $b$ is movable vertically so that the spacings $d_1$ and $d_2$ can be varied within the gap '$a$' of the outer plates. The area ($A$) of each capacitor plate is the same. Initially, the switch $S_1$ is closed and $S_2$ open in order to bring the potential difference of the outer capacitor plates to $V_0$.

Express your answers to the following questions in terms of the variables given in the problem statement above.




(a) Find an expression for the capacitance $C$ of the series combination of the two capacitors and show that it is independent of position of the center conductor. [2pts]

(b) What is the change density $\sigma$ on the top plate when the capacitor is fully charged? [2pts]

(c) What is the change in the energy stored in the capacitor if the center section is removed? [2pts]

(d) Suppose that at the instant the center section is removed, we also remove the battery by opening $S_1$ and closing $S_2$. How long does it take for the voltage on the outer capacitor plates to drop to $V_0/2$? [2pts]


*Solution:* From Gauss's law, we know that, at $t = 0$, in the region between the conductors the electric field is

$$\vec{E}(r) = \frac{Q_0}{r^2}\hat{r} .$$

From Ohm's Law:

$$\vec{J}(r) = \sigma \vec{E}(r) = \frac{\sigma Q_0}{r^2}\hat{r} .$$

To find the current we may integrate the flux of $\vec{J}$ through any spherical surface between the conductors:

$$I = \oint \vec{J} \cdot d\vec{a} = 4\pi r^2 \frac{\sigma Q_0}{r^2} = \underline{4\pi\sigma Q_0} .$$

**(b) (1 point)** Use the result for $I$ from part (a) and Ohm's Law to obtain an expression for the total resistance $R$ between the two conductors in terms of $\sigma$ and $C$.

*Solution:* By Ohm's Law, $R = V/I$, where $V$ is the difference in the electric potential $\varphi$ between the conductors in the capacitor. By the definition of the capacitance, $V = Q/C$. Therefore:

$$R = \frac{Q_0}{C} \cdot \frac{1}{4\pi Q_0 \sigma} = \underline{\frac{1}{4\pi\sigma C}} .$$

**(c) (1 point)** Draw an equivalent circuit for the capacitor/resistor combination.

*Solution:* See Figure 4.

**(d) (2 points)** Write a differential equation for the charge on the capacitor.

*Solution:* By Ohm's Law and the definition of the capacitance:

$$IR = V = \frac{Q}{C} ,$$

where $I$ is the current through the resistor. This current must be equal to minus the time derivative of the charge $Q$ stored in the capacitor. Therefore:

$$-\frac{dQ}{dt} = \frac{Q}{RC} = 4\pi\sigma Q ,$$

with the initial condition that $Q(t = 0) = Q_0$.

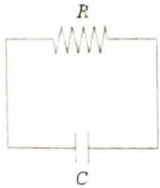



Figure 4: The system is equivalent to a resistance $R$ and a capacitance $C$ connected together. The capacitor will discharge through the resistor.

(e) **(2 points)** Solve for the charge on the capacitor as a function of time in terms of $Q_0$ and $\sigma$.

*Solution:* As explained in Chapter 4 of the textbook, the solution is

$$Q(t) = Q_0\, e^{-4\pi\sigma t} .$$

(f) **(1 point)** What is the total energy deposited in the resistive material between the conductors, in terms of $Q_0$ and $C$ ?

*Solution:* As discussed in chapter 3 of the text:

$$U = \frac{1}{2} C V_0^2 = \underline{\frac{Q_0^2}{2C}} .$$

(g) **(1 point)** The energy deposited per unit volume varies as a function of radius $r$ between the conductors. Derive an expression for the energy deposited per unit volume $\mathcal{E}$ in terms of $Q_0$ and the radial coordinate $r$.

*Solution:* Consider a spherical shell of resistive material, with radius $r$ and thickness $\delta r$. From part **(a)** we know that the current passing through it must be $I(t) = 4\pi\sigma Q(t)$. The resistance of the shell is inversely proportional to the conductance $\sigma$, inversely proportional to the cross-section $A = 4\pi r^2$, and proportional to the thickness $\delta r$. That is:

$$R = \frac{\delta r}{4\pi r^2 \sigma} .$$

The energy per unit time (power) dissipated by the resistive shell is

$$P(t) = \frac{dU}{dt} = I^2(t)\, R = [4\pi\sigma Q(t)]^2 \cdot \left(\frac{\delta r}{4\pi r^2 \sigma}\right) = \frac{4\pi\sigma Q^2(t)\, \delta r}{r^2} ,$$

which implies that the total energy dissipated during the discharge is:

$$U = \int_0^\infty dt\, \frac{4\pi\sigma Q^2(t)\, \delta r}{r^2} .$$

To obtain the energy volume density $\mathcal{E}$ we simply divide this by the volume of the spherical shell $V = 4\pi r^2\,\delta r$, which gives

$$\mathcal{E} = \int_0^\infty dt\,\frac{\sigma Q^2(t)}{r^4}\ .$$

Now we may plug in the result for $Q(t)$ from part (e). The integration then gives:

$$\mathcal{E} = \frac{\sigma Q_0^2}{r^4}\int_0^\infty dt\,e^{-8\pi\sigma t} = \frac{\sigma Q_0^2}{r^4}\frac{-1}{8\pi\sigma}\left[e^{-8\pi\sigma t}\right]_0^\infty = \underline{\frac{Q_0^2}{8\pi r^4}}\ .$$

Another solution to this problem relies on the observation that, since no energy flows radially during the discharge, all of the energy deposited in a given shell of radius $r$ must have originally been contained in that shell's electrostatic field. The volume density of the energy stored in any electrostatic field is:

$$\mathcal{E} = \frac{1}{8\pi}E^2\ .$$

Inside the capacitor the electric field is initially given by:

$$\vec{E}(r) = \frac{Q_0}{r^2}\hat{r}\ ,$$

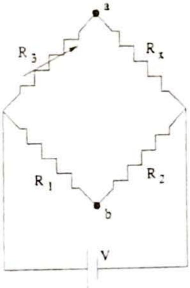
which implies that

$$E^2(r) = \vec{E}(r)\cdot\vec{E}(r) = \frac{Q_0^2}{r^4}\ .$$

Therefore:

$$\mathcal{E} = \underline{\frac{Q_0^2}{8\pi r^4}}\ .$$

A Wheatstone bridge is a device to measure the resistance of an unknown resistor $R_x$ in terms of three known resistances: $R_1$, $R_2$, and a variable resistor $R_3$. Such a device is shown in the diagram below, to which you can refer in answering the following questions.




(a) **(2 points)** What is the voltage $V_{ab}$ between points $a$ and $b$ in terms of the applied voltage $V$ and the 4 resistances?

*Solution:* Applying Ohm's law to each branch, we find the currents $I_{12}$ and $I_{3x}$.

$$
\begin{aligned}
V &= IR_{\text{eff}} \\
I_{12} &= \frac{V}{R_1 + R_2} \quad \text{and} \quad I_{3x} = \frac{V}{R_3 + R_x} \\
V_{ab} &= V_a - V_b \\
&= (V - \Delta V_3) - (V - \Delta V_1) \\
&= \Delta V_1 - \Delta V_3 \\
&= I_{12}R_1 - I_{3x}R_3 \\
&= \underline{V\left(\frac{R_1}{R_1 + R_2} - \frac{R_3}{R_3 + R_x}\right)}
\end{aligned}
$$

(b) **(2 points)** The variable resistor $R_3$ is now adjusted so that $V_{ab} = 0$. Solve for the unknown resistance $R_x$ in terms of the 3 known values $R_1$, $R_2$, and $R_3$.

*Solution:*

$$
\begin{aligned}
V_{ab} &= 0 \\
\Rightarrow \quad \frac{R_1}{R_1 + R_2} &= \frac{R_3}{R_3 + R_x} \\
R_x &= \underline{\frac{R_2 R_3}{R_1}}
\end{aligned}
$$

Assume that $R_1 = 10\Omega$, $R_2 = 20\Omega$, $R_3 = 30\Omega$, and $R_x = 60\Omega$. The EMF source ($V$) is disconnected and a capacitor is connected in its place. The capacitor is initially charged to a voltage of 1000 Volts. The capacitor voltage then decreases to 500 V in a time $10^{-4}$ seconds after being connected to the circuit. Please give numerical

*Solution:* First we find the $R_{\text{eff}}$ for the circuit. Using the series and parallel addition rules, we get:

$$\frac{1}{R_{\text{eff}}} = \frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_x}$$
$$R_{\text{eff}} = 22.5\ \Omega$$

Since we know the solution to the $RC$ circuit, we know the voltage obeys

$$V(t) = V_0 e^{-t_1/(RC)}$$

Given we know the time $t_1$ for the voltage to drop by half, we solve for $C$.

$$\frac{1}{2} = e^{-t_1/(RC)}$$
$$\underline{C = \frac{t_1}{R \ln 2} = 6.4 \times 10^{-6}\ \text{F}}$$

(d) **(2 points)** What is the total energy in Joules that has been dissipated in the resistors during the $10^{-4}$ second time interval?

*Solution:* All the energy lost in resistors has come from the energy stored in the capacitor.

$$U_{\text{lost}} = U_i - U_f = \frac{1}{2}CV_0^2 - \frac{1}{2}CV(t_1)^2$$
$$= \frac{1}{2}CV_0^2\left(1 - \frac{1}{4}\right)$$
$$\underline{= \frac{3}{8}CV_0^2 = 2.4\ \text{J}}$$

(e) **(2 points)** Compute the fraction of the total energy that is deposited in each of the 4 resistors.

*Solution:* Since the potential difference across the branch 12 and $3x$ is always the same no matter how it drops with time, we know about the energy dissipated ratio $E_{12}/E_{3x}$ from $P = V^2/R$,

$$\frac{E_{12}}{E_{3x}} = \frac{R_{3x}}{R_{12}}$$
$$= \frac{30 + 60}{10 + 20}$$
$$= 3$$

Similarly by considering branch 12 and branch 3x individually with $P = I^2R$,

$$\begin{aligned}
\frac{E_2}{E_1} &= \frac{I_{12}^2 R_2}{I_{12}^2 R_1} \\
&= \frac{R_2}{R_1} \\
&= 2 \\
\frac{E_x}{E_3} &= \frac{I_{3x}^2 R_x}{I_{3x}^2 R_3} \\
&= \frac{R_x}{R_3} \\
&= 2
\end{aligned}$$

Therefore, we can compute the fraction of the total energy deposited,

$$\left(\frac{E_1}{E_{\text{total}}}, \frac{E_2}{E_{\text{total}}}, \frac{E_3}{E_{\text{total}}}, \frac{E_x}{E_{\text{total}}}\right) = \left(\frac{3}{12}, \frac{6}{12}, \frac{1}{12}, \frac{2}{12}\right)$$

Problem 5 Solution:

a) If the current is time independent the charges on the capacitors are constant and no current flows to them. We can then forget about the capacitors for a second and consider the circuit consisting only of the EMF going to S1 to R1 to S2 to R2 and back to the EMF. A constant current I flows through this circuit. The power across the 1st resistor is 4W. We can deduce that

$$P = IV = I^2R = I^2 * (2 * 10^3) \to I = \sqrt{\frac{4}{2 * 10^3}} = .045\text{A}. \tag{1}$$

b) Denote the charges on the three capacitors $Q_1, Q_2, Q_3$. We can use Kirchoff loops to calculate the charge on each capacitor. S1 is closed so the kirchoff loop around S1 and C2 implies no charge can exist on C2 since $V_2 = \frac{Q_2}{C_2} = 0$. So

$$Q_2 = 0. \tag{2}$$

Making a Kirchoff loop around $C_1, C_2$ and $R_1$ gives

$$\frac{Q_1}{C_1} - IR_1 = 0 \tag{3}$$

which implies

$$Q_1 = IC_1R_1 = .045(2 * 10^{-6})(2 * 10^3) = 1.79 * 10^{-4}\text{C}. \tag{4}$$

To find $Q_3$ we make a kirchoff loop around the lower box. Following the procedure for $Q_1$ we get

$$Q_3 = IC_3R_2 = .045(4 * 10^{-6})(9 * 10^3) = 1.609 * 10^{-3}\text{C}. \tag{5}$$

c) Denote the emf voltage by $\epsilon$. We can make a loop following the current path to get

$$\epsilon - IR_1 - IR_2 = 0 \to \epsilon = I(R_1 + R_2) = .045 * (2 + 9) * 10^3 = 491.7V. \tag{6}$$

d) Note that if S2 is open all of the current passing through $R_1$ has to pass either to or from $C_3$. We can then formulate a differential equation for this current using Kirchoff's loop law:

$$\epsilon - iR_1 - \frac{Q}{C_3} = 0 \tag{7}$$

where the current i = $\frac{dQ}{dt}$ is now time dependent. Differentiating 7 we find

$$\frac{di}{dt} = -\frac{i}{R_1 C_3}. \tag{8}$$

The solutions to this is

$$i = i_0 e^{-\frac{t}{R_1 C_3}} \tag{9}$$

where $i_0$ = I from part a). We can immediately see that at time

$$t = R_1 C_3 = (2 * 10^3) * (4 * 10^{-6}) = 8 * 10^{-3} s \tag{10}$$

the current reduces to $\frac{1}{e}$ of its original value.
e) We can calculate the charge on $C_3$ before S1 is opened by making a kirchoff loop passing from the EMF to R1 to $C_3$ and back to the EMF:

$$\epsilon - (0) * R_1 - \frac{Q_3}{C_3} = 0 \rightarrow Q_3 = \epsilon C_3 = 491.7 * 4 * 10^{-6} = 1.9668 * 10^{-3} C. \tag{11}$$

If no current is flowing between $C_2$ and $C_3$ the resistor plays no role and can be eliminated from the analysis. By Kirchoff's loop law

$$\epsilon - \frac{Q_2}{C_2} - \frac{Q_3}{C_3} = 0. \tag{12}$$

This implies

$$\frac{Q_2}{6 * 10^{-6}} + \frac{Q_3}{4 * 10^{-6}} = 491.7. \tag{13}$$

Note that when S1 is opened the total charge that is on the negative plate of $C_2$ and the positive plate of $C_3$ has to sum to the charge on the positive plate of $C_3$ right before S1 was flipped by charge conservation (now that the switch is flipped, the charge has nowhere else to go). We then have

$$-Q_2 + Q_3 = 1.9668 * 10^{-3}. \tag{14}$$

This system of equations can be solved to find

$$Q_2 = 0 \text{ and } Q_3 = 1.967 * 10^{-3}\text{C}. \tag{15}$$

The charge across $C_1$ is easy to find by realizing the current through $R_2$ is zero and by using a Kirchoff loop:

$$Q_1 = C_1 V_1 = C_1 * \epsilon = 2 * 10^{-6} * 491.7 = 9.834 * 10^{-4}\text{C}. \tag{16}$$