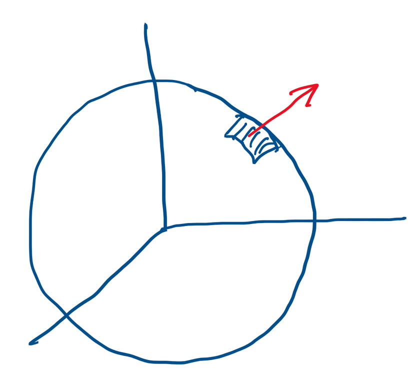
Lecture 11

Monday, 10 February 2020 10:00 AM

To quickly discuss what we should have ideally covered in the last class, let us first define a surface integral.



$d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

When the area element is not taken as part of a closed surface either direction can be regarded as positive.

When the area element is taken as part of a closed surface, the outward direction is positive

We will follow this convention in Gauss's law, which states

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V \rho \, dv / \varepsilon_0$$

(We will see later that this is the integral form of one of the Maxwell equations)

$\int_V \rho \, dv \rightarrow$ enclosed charge

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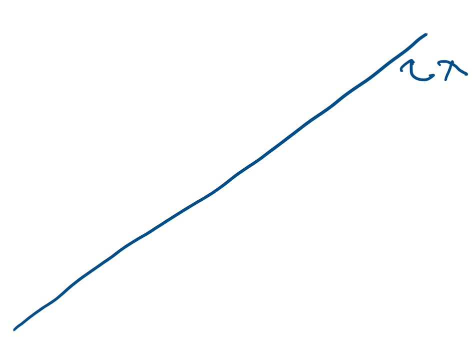


Step 1

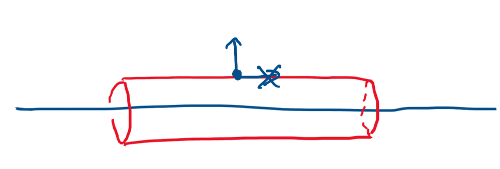
Find a surface where $|\vec{E}|$ is same and construct a Gaussian surface.

Step 2

Use symmetry to eliminate components.

Step 3

Calculate



$$\oint \vec{E} \cdot d\vec{a} = \frac{\int \rho \, dV}{\varepsilon_0}$$

$$\Rightarrow E \times 2\pi r l = \frac{\lambda l}{\varepsilon_0}$$

$$\Rightarrow E = \frac{\lambda}{2\pi r \varepsilon_0} \hat{r}$$

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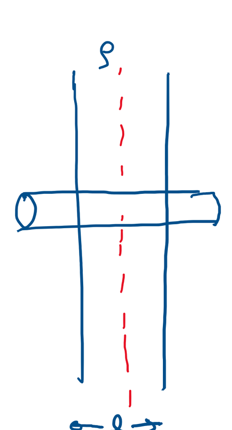

$$\oint \vec{E} \cdot d\vec{a} = \int \rho \, dv / \varepsilon_0$$

$$\Rightarrow 2EA = \rho A l / \varepsilon_0$$

$$\Rightarrow \vec{E} = \rho l / 2\varepsilon_0 \, \hat{z}$$

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$$\vec{E} = \sigma / 2\varepsilon_0$$

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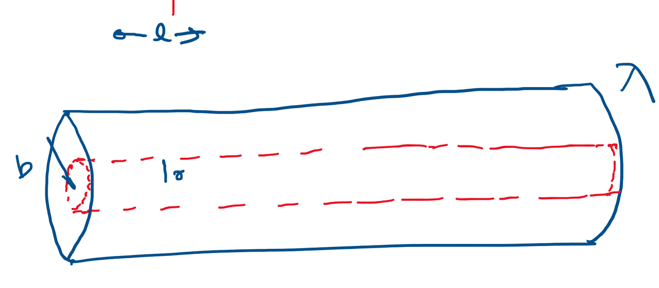


For $r < b$

$$\oint \vec{E} \cdot d\vec{a} = \frac{\int \rho \, dV}{\varepsilon_0}$$

$$\Rightarrow E \times 2\pi r l = 0$$

$$\Rightarrow E = 0$$

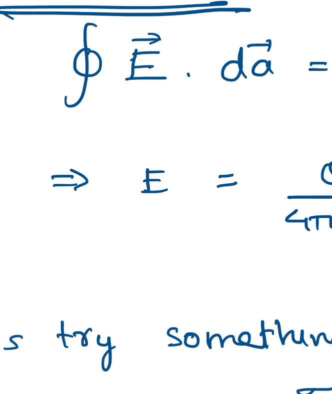
For $r > b$

$$\oint \vec{E} \cdot d\vec{a} = \frac{\int \rho \, dv}{\varepsilon_0}$$

$$\Rightarrow E \times 2\pi r l = \lambda l / \varepsilon_0$$

$$\Rightarrow \vec{E} = \lambda / 2\pi r \varepsilon_0 \, \hat{r}$$

⊛



For $r < R$

$$\oint \vec{E} \cdot d\vec{a} = \int \frac{\rho \, dV}{\varepsilon_0}$$

$$\Rightarrow E \, 4\pi r^2 = \rho \times \tfrac{4}{3} \pi r^3$$

$$\Rightarrow E = \frac{\rho \vec{r}}{3\varepsilon_0}$$

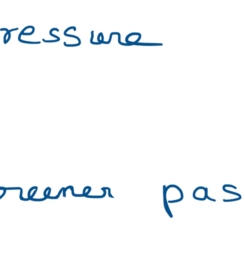
link to 1.17

For $r > R$

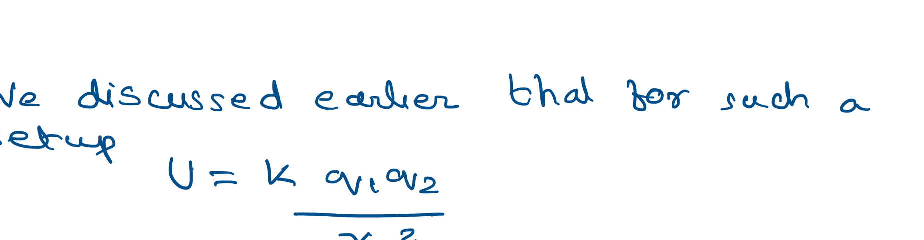
$$\oint \vec{E} \cdot d\vec{a} = \int \rho \, dV / \varepsilon_0$$

$$\Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Let's try something different with it.



Find the pressure due to electric forces trying to expand the shell.



$\therefore$ Force experienced $= \frac{\sigma}{2\varepsilon_0} \times \sigma A$

$\therefore$ Pressure $= \sigma^2 / 2\varepsilon_0$

On to greener pastures

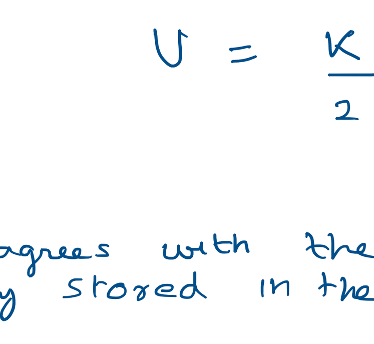
$q_1 \qquad q_2$

We discussed earlier that for such a setup

$$U = k \frac{q_1 q_2}{r_{12}}$$

this energy is stored in the $\vec{E}$ field

What is the energy required to build Q charge on a conducting shell?



$\overleftarrow{dq}$

$$dU = \frac{k q \, dq}{R}$$

$$U = \frac{k q^2}{2R} = \frac{q^2}{8\pi\varepsilon_0 R}$$

This agrees with the result that Dave derived as the energy stored in the electric field.

$\therefore$ Energy Density $= \frac{1}{2} \varepsilon_0 E^2$

$\therefore$ Total Energy $= \int_V \frac{1}{2} \varepsilon_0 E^2 \, dV$

Alternately $U = \frac{1}{2} \int \phi \rho \, dv$

Electric Potential

Earlier we discussed when studying gradient

$$\int_b^a \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$$

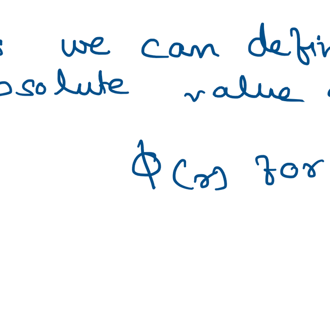
So if we can describe $\vec{E}$ as a gradient of a scalar function, i.e.

$$\vec{E} = -\vec{\nabla}\phi$$

$$\therefore \Delta\phi = -\int \vec{E} \cdot d\vec{s}$$

$\Delta\phi = 0$ when ds is $\perp$ to $\vec{E}$

$\therefore$



Similarly, if $\Delta\phi > 0$, $d\vec{s}$ is against the direction of $\vec{E}$

if $\Delta\phi < 0$, $d\vec{s}$ is in the direction of $\vec{E}$

As we can define $\Delta\phi$, we use $\phi(\infty) = 0$ to define a absolute value of $\phi(r)$.

$$\therefore \phi(r) \text{ for a single particle} = -\int_\infty^r \vec{E} \cdot d\vec{s}$$

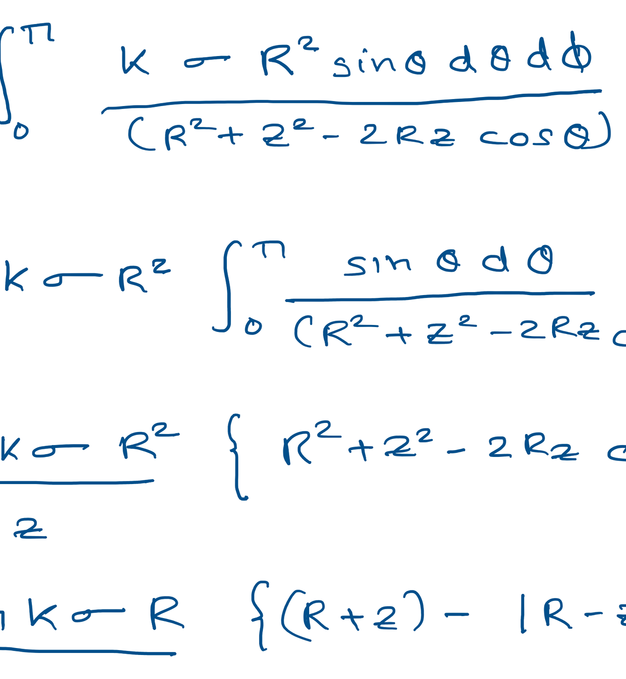
$$= kq/r$$

However, if there are charges at infinity, we can no longer define them in this manner.

For a localized distribution of charge.

$$\Phi = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r})' \, dv'}{|\vec{r} - \vec{r}'|}$$

⊛ Find the potential for a spherical shell using the above formula?



$$V = \int_0^{2\pi} \int_0^{\pi} \frac{k \sigma R^2 \sin\theta \, d\theta \, d\phi}{(R^2 + z^2 - 2Rz\cos\theta)}$$

$$= 2\pi k \sigma R^2 \int_0^{\pi} \frac{\sin\theta \, d\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{1/2}}$$

$$= \frac{2\pi k \sigma R^2}{Rz} \left\{ R^2 + z^2 - 2Rz\cos\theta \right\} \Big|_{\theta=0}^{\theta=\pi}$$

$$= \frac{2\pi k \sigma R}{z} \left\{ (R+z) - |R-z| \right\}$$

for $R < z$

$$V = \frac{4\pi k \sigma R^2}{z} = \frac{kQ}{z}$$

for $R > z$

$$V = \frac{2\pi k \sigma R \, 2\cancel{z}}{\cancel{z}}$$

$$= 4\pi R \sigma k$$

$$= kQ/R$$