

Lecture 3

In the last class, we reasoned out that $\vec{\nabla} \cdot \vec{B} = 0$. We shall spend a minute assessing what that means.

In Electrostatics, we studied that

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0, \text{ which implies that charges can exist in}$$

isolation or excess in space. If that were not true, ρ would be zero always. In other words, charges would always need to exist as dipoles.

$\therefore \vec{\nabla} \cdot \vec{B} = 0$ implies that magnetic monopoles do not exist freely (highly active field of research). Instead, we only encounter magnetic dipoles.

Vector Potential

In electrostatics, we had $\vec{\nabla} \times \vec{E} = 0$ \therefore we could express

$$\vec{E} = -\vec{\nabla} \phi.$$

However ~~Similarly~~, in magnetostatics, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ can be used to describe \vec{B} in terms of a scalar potential only when $\vec{J} = 0$.

Instead, $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ as divergence of a curl is always zero.

A slightly more involved discussion (not covered in class)

$$\rightarrow \vec{\nabla} \times \vec{E} = 0 \Rightarrow \oint \vec{E} \cdot d\vec{l} = 0$$

$\therefore \int_a^b \vec{E} \cdot d\vec{l}$ is path independent scalar quantity.

We define this quantity as $\Delta \phi_{ab} = - \int_a^b \vec{E} \cdot d\vec{l}$.

$$\rightarrow \vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \oint \vec{B} \cdot d\vec{a} = 0$$

$\therefore \int_S \vec{B} \cdot d\vec{a}$ is dependent only on the curve bounding the surface S . The actual surface does not matter.

∴ To describe \vec{B} in terms of a field that can be calculated along the bounding curve.

$$\therefore \int_S \vec{B} \cdot d\vec{a} = \int_S (\nabla \times \vec{A}) \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{l}$$

$$\therefore \vec{B} = \nabla \times \vec{A}$$

— x — x — x —

(I am about to rush through the rest. Stop me where you feel I am too quick).

Can we obtain a unique \vec{A} for a given \vec{B} ?

The answer seems to be 'No!'

For example, let us take $\vec{B} = \nabla \times \vec{A}$

Now define $\vec{A}' = \vec{A} + \nabla \phi$

$$\therefore \nabla \times \vec{A}' = \nabla \times (\vec{A} + \nabla \phi) = \nabla \times \vec{A} = \vec{B}$$

∴ magnetic fields, like electric fields, exhibit gauge invariance.

$$\vec{A} \rightarrow \vec{A} + \nabla \phi$$

$$\phi \rightarrow \phi + c$$

~~We also~~

Further,

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

To make life easier and obtain an expression of \vec{A} in terms of \vec{J} ; we impose $\nabla \cdot \vec{A} = 0$ (explain here why that is allowed)

$$\Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Using Green's function, $\vec{A}(\vec{r}_1) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|} dV(\vec{r}_2)$

& from Dave's Notes,

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times (\vec{r}_1 - \vec{r}_2)}{(\vec{r}_1 - \vec{r}_2)^3}$$

(I have avoided the derivation here as it is merely mathematics)

~~line~~ line current

$$\vec{A}(\vec{r}_1) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}_2) dV(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|}$$

surface current

$$= \frac{\mu_0}{4\pi} \int \frac{\vec{K} da(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|}$$

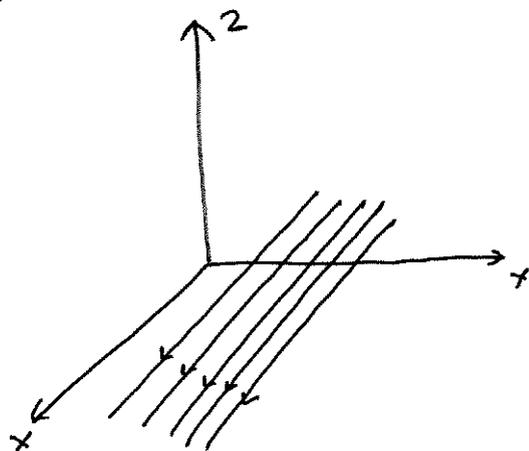
$$\vec{B}(\vec{r}_1) = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3} dV(\vec{r}_2)$$

$$= \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3} da(\vec{r}_2)$$

$$= \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$$

Problem 1

Find the magnetic field of an infinite uniform surface current $\vec{K} = k \hat{x}$, flowing over the x - y plane

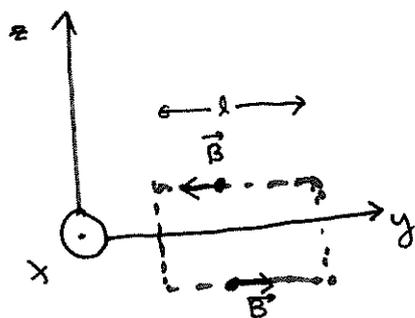


Let us first find B_x, B_y, B_z qualitatively for a general point in space

$B_x \rightarrow$ cannot be non zero (From Biot Savart's law)

$B_z \rightarrow$ cannot be non zero. Contributions by wire segments to the left (along y) of this point cancel out the contributions by wire segments to the right (along y) of this point.

$B_y \rightarrow$ We are now at a stage to implement Ampere's law



(Using Biot-Savart's law to determine direction)

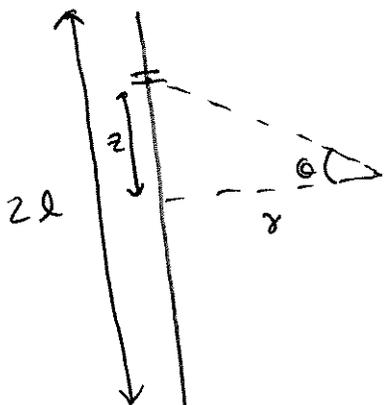
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\Rightarrow 2Bl = \mu_0 k l$$

$$\Rightarrow B = \begin{cases} \frac{\mu_0 k}{2} \hat{y} & , \text{ for } z < 0 \\ -\frac{\mu_0 k}{2} \hat{y} & , \text{ for } z > 0 \end{cases}$$

Problem 2

Find the vector potential of a thin infinite wire



$$\vec{A} = \frac{\mu_0 I}{2\pi} \int_{-l}^l \frac{d\vec{l}}{|\vec{z} - \vec{r}'|}$$

$$= \frac{\mu_0 I}{2\pi} \hat{z} \int \frac{dz}{(z^2 + r^2)^{3/2}}$$

$$= \frac{\mu_0 I}{2\pi} \ln \left(\frac{\sec \alpha + \tan \alpha}{\sec(-\alpha) + \tan(-\alpha)} \right) \hat{z}$$

$$= \frac{\mu_0 I}{2\pi} \ln \left(\frac{\sqrt{z^2 + l^2} + l}{\sqrt{z^2 + l^2} - l} \right) \hat{z}$$

Find \vec{B}

As $l \rightarrow \infty$

$$= \frac{\mu_0 I}{2\pi} \ln \left(\frac{2l}{r} \right)$$

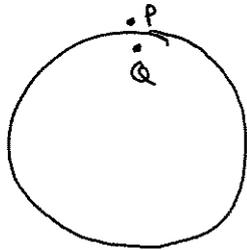
\vec{A} can have some very different characteristics. It doesn't need to go to zero to infinity.

Find \vec{B} for this \vec{A} .

- Discuss the pitfalls of taking a curl of the final result here.
- It is always better to take the curl of the general result and then take $l \rightarrow \infty$.

Pressure on a current carrying hollow cylinder with surface current density $k \odot$

$z \odot$
 $k \odot$



For the cylinder,

$\vec{B} = 0$ at Q using Ampere's law

At P,

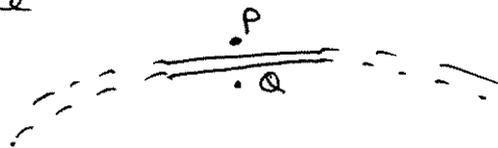
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\Rightarrow B_\phi \times 2\pi R = \mu_0 k 2\pi R$$

$$\Rightarrow \vec{B}_\phi = \mu_0 k$$

$$\therefore \vec{B} = \begin{cases} \mu_0 k \hat{e}_\phi & \text{at P} \\ 0 \hat{e}_\phi & \text{at Q} \end{cases}$$

For a small strip on the cylinder, where P is very close to the surface



$$\vec{B} = \begin{cases} \frac{\mu_0 k}{2} \hat{e}_\phi, & \text{at P} \\ -\frac{\mu_0 k}{2} \hat{e}_\phi, & \text{at Q} \end{cases}$$

\therefore By superposition, the \vec{B} due to cylinder in the absence of the strip = $\frac{\mu_0 k}{2} \hat{e}_\phi$ at P & Q

\therefore Force on the strip = $I (\vec{l} \times \vec{B}) = -k A \frac{\mu_0 k}{2} \hat{z}$ (where $I = k \delta \leftarrow$ width of st & $\delta l = A \leftarrow$ area of st)

\therefore Pressure = $\frac{\mu_0 k^2}{2}$ (inwards)

Extra - Notes

In the class, we derived

$$\vec{A} = \frac{\mu_0 I}{2\pi} \ln\left(\frac{2l}{r}\right) \hat{z} \quad \text{for a infinite wire}$$

(+ve \hat{z} direction) where r is greater than the radius of the wire.

On the other hand, Purcell derives,

$$\vec{A} = -\frac{\mu_0 I}{2\pi} \ln(r) \hat{z}$$

(-ve \hat{z} direction).

Both answers are correct, they are related by gauge invariance where $\vec{\nabla} f = \frac{\mu_0 I}{2\pi} \ln(2l)$

Similarly, for inside the wire

$$\vec{A} = -\frac{\mu_0 I}{4\pi R^2} (r^2 - b^2) \hat{z}$$

arbitrary constant.

(can be +ve or -ve based on b)

∴ While we can ensure \vec{A} must be parallel to \vec{J} , the direction of \vec{A} could be either +ve or -ve.

When, solving both inside and outside the cylinder together, just be sure to impose continuity.

Discontinuous \vec{A} would imply infinite magnetic field and magnetic monopoles in the region.