Lecture 9

Thursday, 31 October 2019 9:19 AM

Angular Momentum : $\vec{L} = \vec{r} \times \vec{p}$ | $\vec{P} = m\vec{v}$
Fundamental Law : Conserved | Conserved

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}$$

$$= \underbrace{\vec{r} \times \vec{F}}_{\vec{\tau}} \quad \text{(torque)}$$

$$\frac{d\vec{p}}{dt} = \vec{F}$$
second law

When there is no force
$\vec{p}$ is conserved

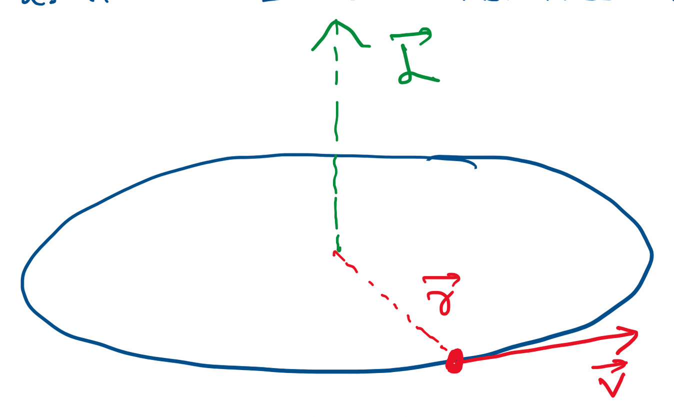
When there is no torque
$\vec{\tau}$ is conserved

Now there are two possible ways $\vec{L}$ could change,
its magnitude and its direction.

For this course, we shall focus on changes in magnitude and effectively treat it as a scalar.

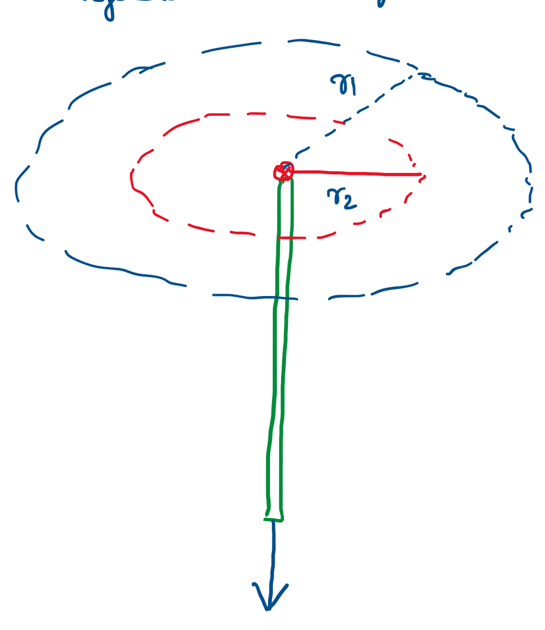
Lets start calculating

1) A particle of mass m is undergoing uniform circular motion with velocity $\vec{v}$. What is the angular momentum as it moves around the circle?



$$|\vec{L}| = |\vec{r} \times \vec{p}|$$
$$= m\,|\vec{r} \times \vec{v}|$$
$$= mrv$$

2) A mass m is twirled in a circle of radius r_1 with constant speed v_1. If the string is pulled so that moves in a circle of radius r_2, what is the new velocity v_2 and angular velocity ω_2?



When the mass is circling at radius r_1, an inwards tension T provides the centripetal force.

We increase the T to pull the particle inward.

$$\therefore \quad \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = 0$$

$$\therefore \quad m r_1 v_1 = m r_2 v_2$$

$$\therefore \quad v_2 = \frac{r_1 v_1}{r_2}$$ (velocity increases as radius decreases)

$$\& \quad \omega_1 = v_1/r_1$$

$$\Rightarrow \omega_2 = \frac{v_2}{r_2} = \frac{r_1}{r_2^2} v_1 = \frac{r_1^2}{r_2^2}\omega_1$$

(the classic example of the ice skater)

3) A cannon shoots a projectile with velocity v_0 at angle θ. Is the angular momentum relative to the cannon conserved? Why?

$$\vec{v} = v_0 \cos\theta\,\hat{i} + (v_0 \sin\theta - gt)\,\hat{j}$$
$$\vec{r} = (v_0 t \cos\theta)\,\hat{i} + (v_0 t \sin\theta - \tfrac{1}{2} g t^2)\,\hat{j}$$

At $t = 0$, $\vec{L} = \vec{r} \times \vec{p} = 0$

At $t = t$, $\vec{L} = m(\vec{r} \times \vec{v})$

$$= m\Big(v_0^2 \sin\theta\cos\theta\, t - \cancel{gt}\, v_0 g t^2 \cos\theta$$
$$- v_0^2 \sin\theta \cos t + \tfrac{1}{2} g t^2 v_0 \cos\theta\Big)\hat{k}$$
$$= -\tfrac{1}{2} m g v_0 \cos\theta\, t^2\, \hat{k}$$

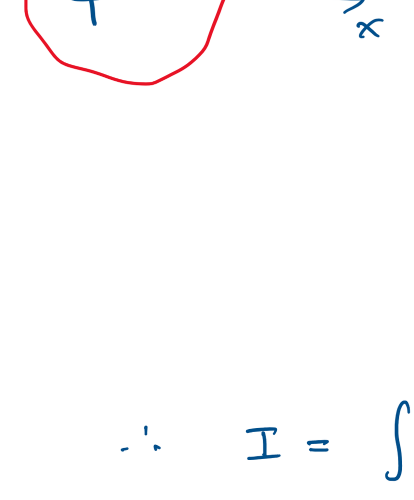
Where does this come from?

$$\tau = \vec{r} \times (-mg\,\hat{j})$$
$$= -mg\cos\theta\, v_0 t\,\hat{k}$$

$$\therefore \quad \int_0^L dL = \int_0^t \tau\, dt$$

$$L = -\frac{mg\cos(\theta)\, t^2}{2}\,\hat{z}$$

Now, lets introduce another concept that makes the study of rotational bodies much easier.



$$\vec{\omega} = \omega\,\hat{z}$$
$$v = \omega r$$
$$\vec{L} = \vec{r} \times \vec{p}$$

$$d\vec{L} = r^2 \omega\, dm\,\hat{z}$$
$$\vec{L} = \int r^2 \omega\, dm\, d\hat{z}$$
$$= \omega \int dm\, r^2\, d\hat{z}$$

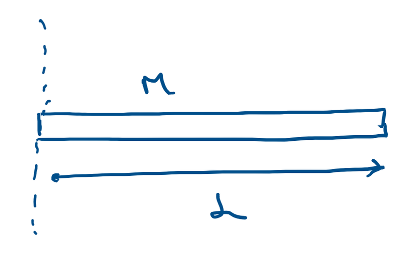
$$\therefore \quad I = \int dm\, r^2$$

$$\& \quad \vec{L} = I\vec{\omega}$$
↑ the rotational equivalent of mass

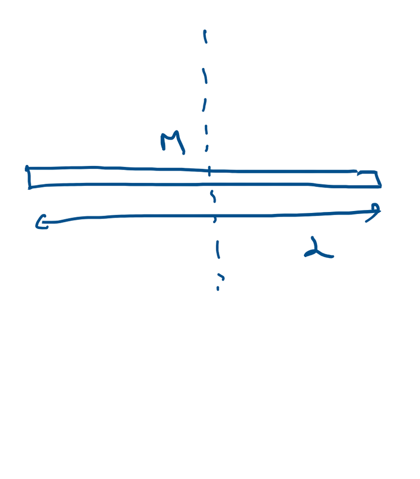
I is defined about an axis unlike $\vec{L}$ & $\vec{\tau}$ which are defined about a point. For this class, that distinction wouldnt matter much.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = I\vec{\alpha}$$

For example,

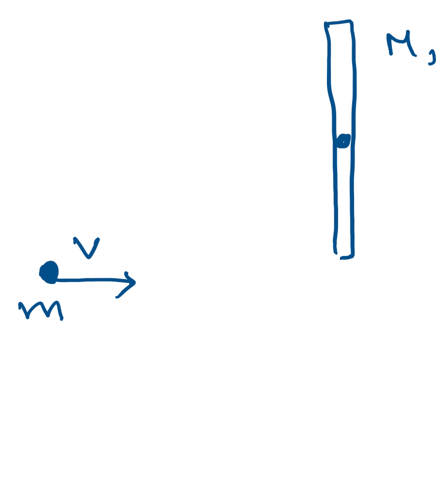


$$I = \int dm\, r^2$$
$$= \int_0^L \frac{M}{L}\, dr\, r^2$$
$$= \frac{ML^2}{3}$$



$$I = \int_{-L/2}^{L/2} \frac{M}{L}\, dr\, r^2$$
$$= \frac{M}{3L}\left[\frac{L^3}{8} + \frac{L^3}{8}\right]$$
$$= \frac{ML^2}{12}$$

4) Now consider a particle is shot towards a rod with velocity v. The rod is hinged about a point. Calculate the 'ω' of the rod if the particle sticks to the rod.



$$mv \times \frac{L}{2} = \left(\frac{ML^2}{12} + \frac{mL^2}{4}\right)\omega$$

$$\Rightarrow \omega = \frac{mv}{\left(\frac{M}{6} + \frac{m}{2}\right)L}$$

$$\Rightarrow \omega = \frac{6m}{(M + 3m)}\,\frac{v}{L}$$