Lecture 7

Wednesday, 23 October 2019 3:36 PM

Let us start with Newton's 2nd Law in 1 Dimension,

$$F(x) = ma$$

$$\Rightarrow F(x) = m v \frac{dv}{dx}$$

Constant for a conservative system

$$\Rightarrow E = \frac{1}{2}mv^2 - \int F(x)dx$$

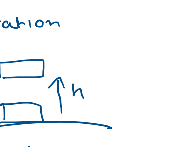
K.E, P.E

$\therefore K = \frac{1}{2}mv^2$ ← associated to the freedom of motion

$\Delta U = -\int_{x_0}^{x} F(x)dx$ ← associated to the freedom of position

There is no absolute definition of potential energy. We need a reference.

Let us see for the case of Gravitation



$$\Delta U = -\int_{\infty}^{r} \frac{GMm}{r^2}dr \qquad \Delta U = \int_0^h mg\,dy = mgh$$

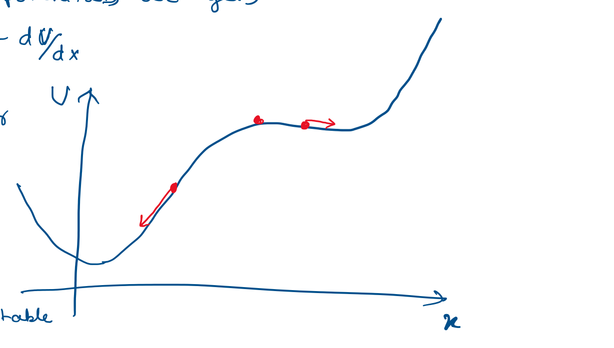
$$U(r) = -\frac{GMm}{r} + 0$$

Reference U at ∞ = 0

Conversely, given any potential we get,

$$F = -\frac{dU}{dx}$$

This means that, for



At extrema, check for,

$dU/dx = 0$, d^2U/dx^2

to find stable & unstable equilibrium

In 3D, $\vec{F} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$

$$= -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)U$$

$$= -\vec{\nabla}(U)$$

← Gradient operator

Gradient provides direction of fastest change and is ⊥ to equipotential lines.

We can also define $\Delta W = \Delta U = \int F(x)dx$ (ΔU is only defined for a conservative system)

In 3D, this becomes $\Delta W = \int F_x dx + \int F_y dy + \int F_z dz$

$$= \int \vec{F}\cdot d\vec{r}$$

An alternate expression, Work Energy Theorem

$$\Delta W = \int \vec{F}\cdot d\vec{r} = \int m\frac{d\vec{v}}{dt}\cdot d\vec{r}$$

$$= \int \frac{1}{2} m\, d(\vec{v}\cdot\vec{v})\, dt$$

$$= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \Delta K$$

(always holds irrespective of conservative or non conservative forces)

∴ For a conservative system,

$$\Delta W = -\Delta U = \Delta K$$

$$\Rightarrow \Delta U + \Delta K = 0 \quad \text{(or energy is conserved)}$$

Additionally we can show that for energy to be conservative only need $dU/dt = 0$

$$\vec{F}\cdot d\vec{r} = d\left(\frac{1}{2}mv^2\right) = dK$$

$$\Rightarrow \frac{dK}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt}$$

$$\Rightarrow dK/dt = \vec{F}\cdot\vec{v}$$

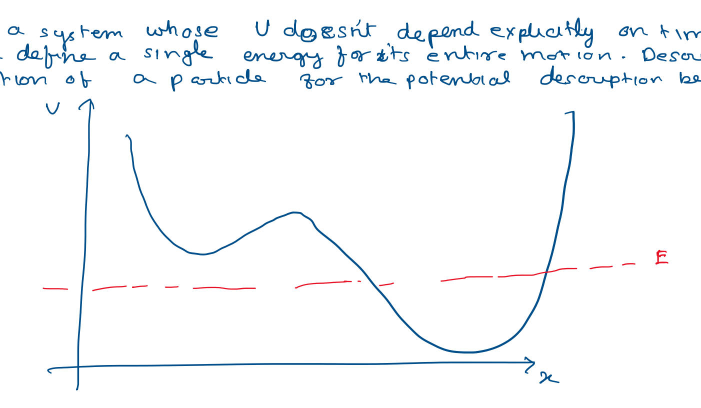
Also, by chain rule, $\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} + \frac{\partial U}{\partial t}$

$$= \left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)\cdot(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}) + \frac{\partial U}{\partial t}$$

$$= -\vec{F}\cdot\vec{v} + \frac{\partial U}{\partial t}$$

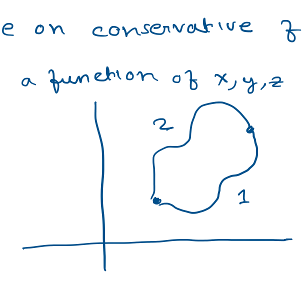
$$\therefore \frac{dE}{dt} = \partial U/\partial t$$

For a system whose U doesn't depend explicitly on time, we can define a single energy for its entire motion. Describe the motion of a particle for the potential description below.



Some more on conservative forces,

if U is a function of x, y, z alone



$$\int_1 \vec{F}\cdot d\vec{s} = \int_2 \vec{F}\cdot d\vec{s}$$

$\therefore \int \vec{F}\cdot d\vec{s}$ is path independent.

$$\oint \vec{F}\cdot d\vec{s} = 0$$

Now $\oint \vec{F}\cdot d\vec{s} = \int(\vec{\nabla}\times\vec{F})\,dA$ (we will discuss this further in Ph 1 b)

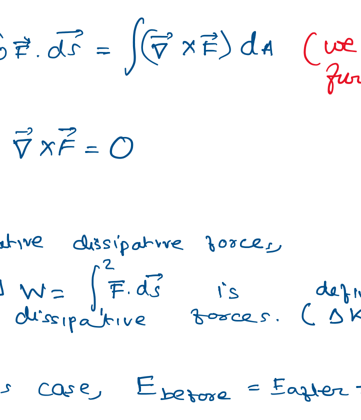
$$\therefore \vec{\nabla}\times\vec{F} = 0$$

For non conservative dissipative forces,

$\Delta W = \int^2 \vec{F}\cdot d\vec{s}$ is defined as the energy lost due to dissipative forces. (ΔK is negative)

In such a case, $E_{before} = E_{after} + W_{dissipated}$.

1) A particle moves toward $x=0$ under the influence of a potential $U = -A|x|^n$, where A & $n > 0$. It has barely enough energy to reach 0. For what values of 'n' will it reach in finite time for a finite x_0 (point of release)



$$E = U(0) = 0$$

By conservation of energy,

$$0 = \frac{1}{2}mv^2 - A|x|^n$$

$$v = -\left(\frac{2A}{m}\right)^{1/2} x^{n/2}$$

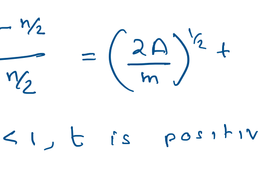
$$\Rightarrow \int_{x_0}^{0} \frac{dx}{x^{n/2}} = -\int_0^t \left(\frac{2A}{m}\right)^{1/2} dt$$

$$\Rightarrow \frac{x_0^{1-n/2}}{1-n/2} = \left(\frac{2A}{m}\right)^{1/2} t$$

For $n/2 < 1$, t is positive and $\therefore n < 2$

2) A marble of mass 'm' is deposited inside a hemispherical bowl of radius 'R', as shown in the figure. The bowl is then spun around its vertical axis with constant angular velocity 'ω'. The marble eventually settles at a radius 'r' from the axis. Is the energy conserved, where does the extra energy come from.

(Discuss in class)

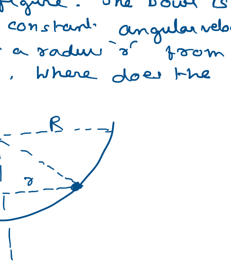


3) A daredevil astronomer stands at the top of an observatory dome wearing roller skates. He starts to coast down over the dome surface with an initial velocity V_0. Neglect friction

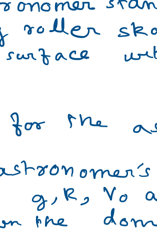
a) Draw a FBD for the astronomer.

b) What is the astronomer's velocity in terms of g, R, V_0 and θ as he rolls down the dome?

c) At what angle θ does the astronomer leave the dome.

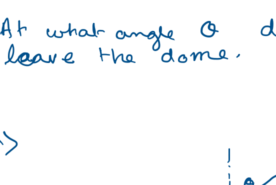


a)



$$N + mv^2/r - mg\cos\theta$$

b)



$$\frac{1}{2}mv^2(\theta) - \frac{1}{2}mv_0^2 = mgR(1-\cos\theta)$$

$$v^2(\theta) = v_0^2 + 2gR(1-\cos\theta)$$

c) $N = mg\cos\theta - \frac{mv^2}{R}$

$$= mg\cos\theta - \frac{mv_0^2}{R} - 2mg(1-\cos\theta)$$

The astronomer leaves the surface when $N = 0$

$$\cos\theta = \frac{V_0^2}{Rg} + 2 - 2\cos\theta$$

$$\Rightarrow \cos\theta = 2/3 + V_0^2/3Rg$$

4) A man of height $h_0 = 2m$ is bungee jumping from a platform at a height of $h = 25m$ above a lake. One end of an elastic rope is attached to the foot and the other end is fixed to the platform.

The length and elastic properties of the rope are chosen so that his speed is zero at the instead his head reaches the surface of the water. Ultimately the jumper hangs from the rope with the head 8m above the water.

i) Find the unstretched length of the rope.
ii) Find the maximum speed and acceleration acheived during the jump.

a) Assume the spring constant of the rope by 'k'. The unstretched length of the rope be l_0.
Maximum length of rope $l_1 = 23$ meters.
Equilibrium length of rope $l_2 = 15$ meters.
And the center of mass for the person is half way in the body.

At equilibrium,

$$mg = k(l_2 - l_0) \quad \dots (i)$$

&

$$mgh = \frac{1}{2}k(l_1 - l_0)^2$$

$$\Rightarrow h = \frac{(l_1 - l_0)^2}{2(l_2 - l_0)}$$

$$\Rightarrow l_0^2 + 2(h - l_1)l_0 + (l_1^2 - 2hl_2) = 0$$

$$l_0 \simeq 13 \text{ meters}$$

b) When the jumper reaches maximum speed, the acceleration must be zero. This occurs at the equilibrium position, $l = l_2$. At this point, the CM has fallen to $l_2 + h_0$.

$$\therefore \frac{1}{2}mv_M^2 + \frac{1}{2}k(l_2 - l_0)^2 = mg(l_2 + h_0)$$

(i) $k = mg/(l_2 - l_0)$

$$\therefore v_M^2 = 2g(l_2 + h_0) - k/m\,(l_2 - l_0)^2$$

$$= 32g$$

$$V_M = 18 \text{ m/s}$$

The maximum force and hence acceleration is at the lowest point of the jump.

$$F_M = k(l_1 - l_0) = 10k$$

$$\& \quad mg = k(l_2 - l_0) = 2k$$

$$\therefore F_M = 5mg$$

$$\therefore \text{Net acceleration} = \frac{5mg - mg}{m} = 4mg$$

$$\therefore \boxed{a_m = 4g} \text{ upwards}$$