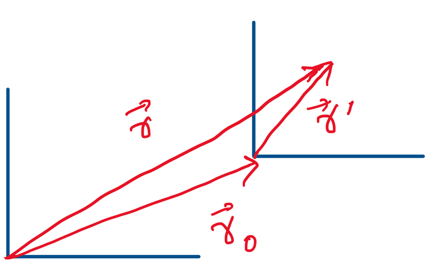
Lecture 6

Monday, 21 October 2019 12:49 AM

In class, Cliff discussed that:



$$\vec{r} = \vec{r}\,' + \vec{r}_0$$

$$\vec{v} = \vec{v}\,' + \vec{v}_0 \rightarrow v_0 = \text{constant (inertial frame)}$$

$$\vec{a} = \vec{a}\,' + \vec{a}_0 \rightarrow a_0 \neq 0 \text{ (non-inertial frame)}$$

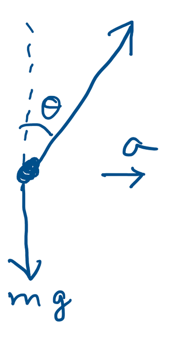
$$\vec{F} = \vec{F}\,' + m\vec{a}_0$$

$$\therefore \vec{F}\,' = \vec{F} - m\vec{a}_0$$

(discuss implications in General Relativity)

1) A small weight of mass 'm' hangs from a string in a car which accelerates at 'a' in the ground frame. What is the static angle of the string from the vertical and the tension?

Inertial Frame



$T\cos\theta - mg = 0$

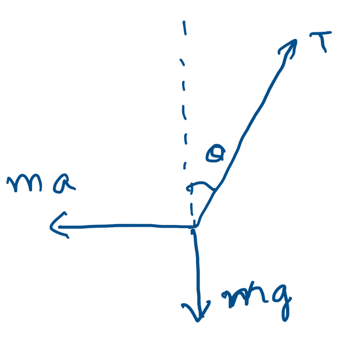
& $T\sin\theta = ma$

$\tan\theta = a/g$

$\therefore T = \dfrac{ma}{\sin\theta} = \dfrac{ma\sqrt{a^2+g^2}}{a}$

$= m(a^2+g^2)^{1/2}$

Non Inertial frame



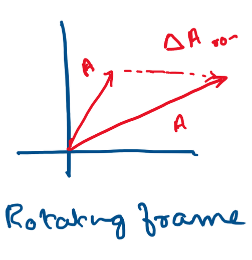
$T\cos\theta - mg = 0$

$T\sin\theta - ma = 0$

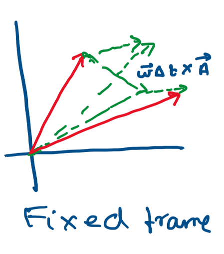
$\tan\theta = a/g$

$\therefore T = m(a^2+g^2)^{1/2}$

Forces in rotating frames (Advanced Section)



Rotating frame



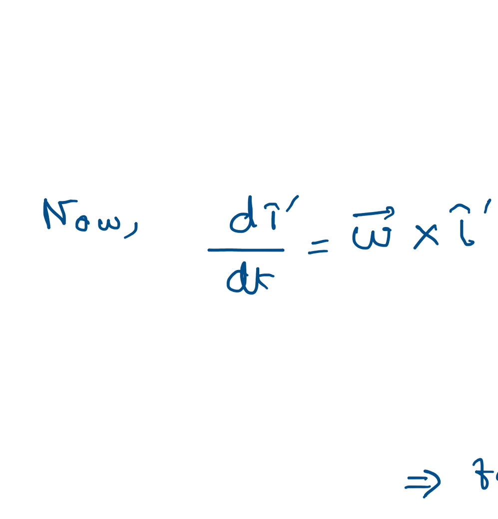
Fixed frame

$$(\Delta\vec{A}) = \Delta\vec{A}_{rot} + \vec{\omega}\Delta t \times \vec{A}$$

$$\Rightarrow \frac{\Delta A}{\Delta t} = \frac{\Delta\vec{A}_{rot}}{\Delta t} + \vec{\omega}\times\vec{A}$$

$$\Rightarrow d\vec{A}/dt = \frac{d\vec{A}_{rot}}{dt} + \vec{\omega}\times\vec{A}$$

$$\Rightarrow \vec{v} = \vec{v}_{rot} + \vec{\omega}\times\vec{r} \text{ or } \vec{\omega}\times\vec{r}\,'$$



Any arbitrary vector B

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\vec{B} = B_x'\hat{i}' + B_y'\hat{j}' + B_z'\hat{k}'$$

$$\frac{\partial B}{\partial t} = \frac{\partial B_x}{\partial t}\hat{i} + \frac{\partial B_y}{\partial t}\hat{j} + \frac{\partial B_z}{\partial t}\hat{k}$$

$$\frac{\partial B}{\partial t} = \left(\frac{\partial B_x'}{\partial t}\hat{i}' + \frac{\partial B_y'}{\partial t}\hat{j}' + \frac{\partial B_z'}{\partial t}\hat{k}'\right) + \left(B_x'\frac{\partial\hat{i}'}{\partial t} + B_y'\frac{\partial\hat{j}'}{\partial t} + B_z'\frac{\partial\hat{k}'}{\partial t}\right)$$

Now, $\dfrac{d\hat{i}'}{dt} = \vec{\omega}\times\hat{i}'$ (as $\vec{v}_{rot}$ of $\hat{i}'$ is zero)

$$\left(\frac{\partial B}{\partial t}\right)_{in} = \left(\frac{\partial B}{\partial t}\right)_{rot} + \vec{\omega}\times\vec{B}$$

$\Rightarrow$ for $\vec{B} = \vec{r}$

$$\vec{v}_{in} = \vec{v}_{rot} + \vec{\omega}\times\vec{r}$$

& for $\vec{B} = \vec{v}_{in}$

$$\left(\frac{\partial\vec{v}_{in}}{\partial t}\right)_{in} = \left(\frac{\partial\vec{v}_{in}}{\partial t}\right)_{rot} + \vec{\omega}\times\vec{v}_{in}$$

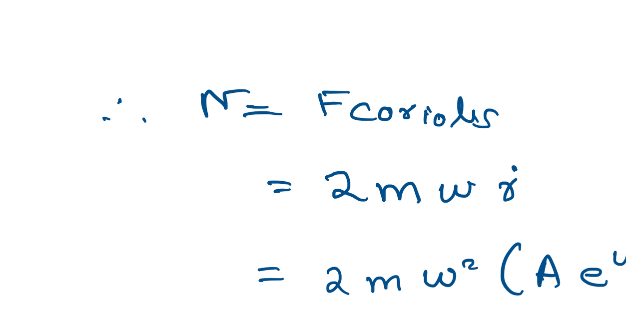
$$\Rightarrow \vec{a}_{in} = \frac{\partial}{\partial t}\left(\vec{v}_{rot} + \vec{\omega}\times\vec{r}\right) + \vec{\omega}\times\left(\vec{v}_{rot} + \vec{\omega}\times\vec{r}\right)$$

$$\Rightarrow \vec{a}_{in} = \vec{a}_{rot} + \vec{\omega}\times\left(\frac{\partial\vec{r}}{\partial t}\right)_{rot} + \vec{\omega}\times\vec{v}_{rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\Rightarrow \vec{a}_{rot} = \vec{a}_{in} - \Big(\underbrace{2\vec{\omega}\times\vec{v}_{rot}}_{\text{Coriolis}} + \underbrace{\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{centripetal/centrifugal}}\Big)$$

2) A bead slides without friction on a rigid wire rotating at constant angular speed 'ω'. Find the force exerted by the wire on the bead and the position as a function of time

In rotational frame



Now, $m\ddot{r} = F_{centrifugal}$ & $N - F_{coriolis} = 0$

$\Rightarrow m\ddot{r} = m\omega^2 r$

$\therefore r = Ae^{\omega t} + Be^{-\omega t}$

$\therefore N = F_{coriolis}$

$= 2m\omega\dot{r}$

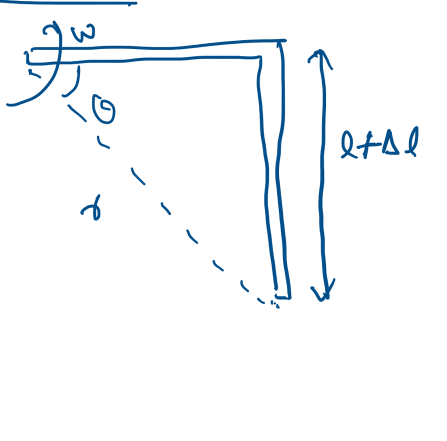
$= 2m\omega^2(Ae^{\omega t} - Be^{-\omega t})$

Assume at $t=0$, $r = a$ & $\dot{r} = 0$

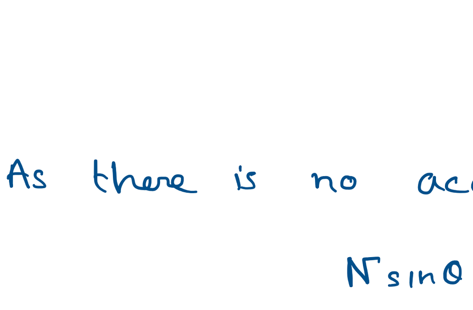
$\therefore r = a\cosh(\omega t)$

In the fixed frame, we need a $F = m\omega^2 r$ centripetal force to hold the bead in place.

3) A device consists of a smooth L-shaped rod located on a horizontal plane and a sleeve of mass 'm' attached by a massless spring to a point B. The spring stiffness is 'k'. The whole system rotates with constant angular velocity 'ω' about a vertical axis through O. Find the elongation of the spring.

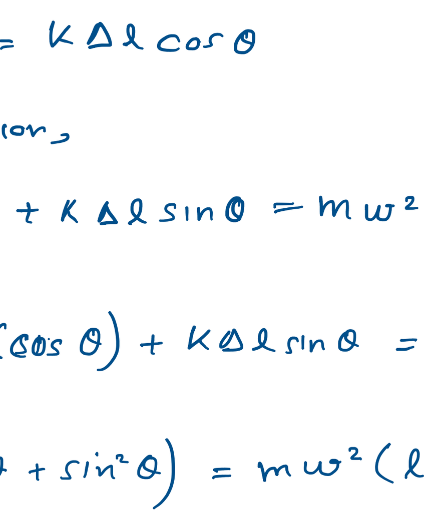


Top view



Spring Force $= k\Delta l$

& N = normal of sleeve & rod



As there is no acceleration in the tangential direction,

$$N\sin\theta = k\Delta l\cos\theta$$

In the radial direction,

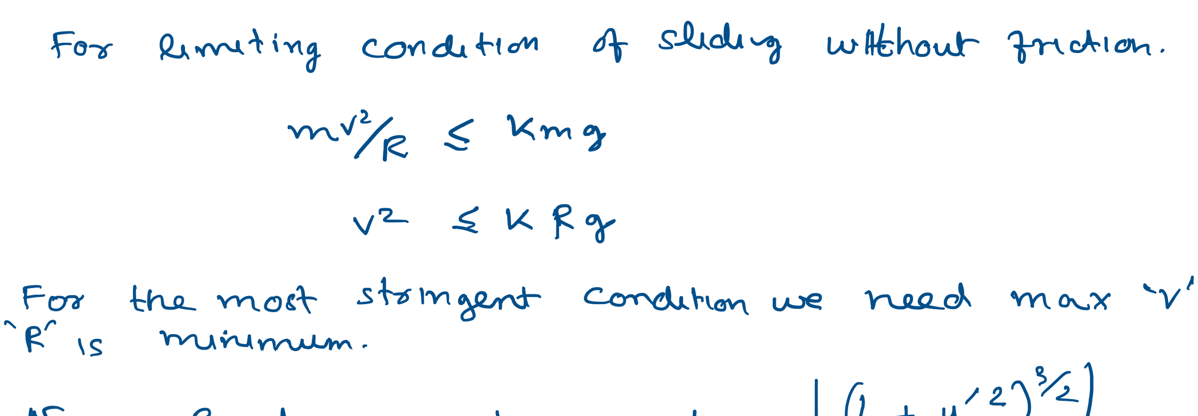
$$N\cos\theta + k\Delta l\sin\theta = m\omega^2 r$$

$$\Rightarrow k\Delta l\frac{\cos\theta}{\sin\theta}(\cos\theta) + k\Delta l\sin\theta = m\omega^2 r$$

$$\Rightarrow k\Delta l\left(\cos^2\theta + \sin^2\theta\right) = m\omega^2(l+\Delta l)$$

$$\Rightarrow \Delta l = \frac{ml\omega^2}{k\Delta l - m\omega^2}$$

5) A car moves uniformly along a horizontal sine curve $y = a\sin(x/\alpha)$. The coefficient of friction between roads & wheels is equal to k. At what velocity can the car ride without sliding



Centripetal force $= mv^2/r$

For limiting condition of sliding without friction.

$$mv^2/R \leq kmg$$

$$v^2 \leq kRg$$

For the most stringent condition we need max 'v' when 'R' is minimum.

Now R at any point is given by $\left|\dfrac{(1+y'^2)^{3/2}}{y''}\right|$

we find that it is minimum when the curve is at its maxima & minima

$\therefore$ at this point the entire centripetal force is along 'y'.

$$\therefore a_y = \frac{d^2y}{dt^2} = -\frac{v^2a}{\alpha^2}\sin(x/\alpha) = -\frac{v^2a}{\alpha^2}$$

At this point $\dfrac{v^2}{R} = -a_y$

$$\Rightarrow R = \frac{v^2\alpha^2}{v^2a} = \alpha^2/a$$

$$\therefore v^2 \leq \frac{k\alpha^2 g}{a}$$

$$\therefore v \leq \alpha\sqrt{\frac{kg}{a}}$$