Lecture 15

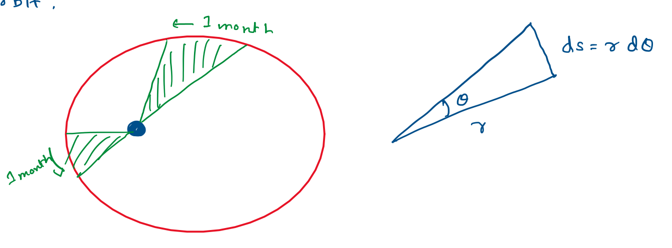
[Clarify E equation]

Wednesday, 20 November 2019 5:41 PM

Keplers laws

1) Planets move in elliptical orbits with sun at one focus.
If a planet is bound it must be in elliptical or circular orbit. The chances of being in a circular orbit are small as energy needs to be exactly zero.

2) The radius vector of a planet sweeps out an area at a rate that is independent of the position in the orbit.



$$\therefore dA = \frac{1}{2} r (r d\theta) = \frac{r^2 d\theta}{2}$$

$$\frac{dA}{dt} = \frac{r^2}{2}\dot{\theta}$$

$$\frac{dA}{dt} = \frac{m r^2 \dot{\theta}}{2m} = \frac{L}{2m} \rightarrow \text{all constants}$$

3) $T^2 \propto a^3$

From above,

$$\frac{dA}{dt} = \frac{L}{2m}$$

$$A = \frac{LT}{2m}$$

Area of ellipse $= \pi a b$
$= \pi a^2 (1-\epsilon^2)^{1/2}$

$$\therefore \pi^2 a^4 (1-\epsilon^2) = \frac{L^2 T^2}{4m^2}$$

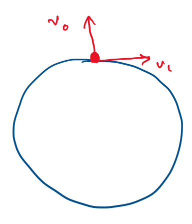
$$\Rightarrow T^2 = \frac{4 m^2 \pi^2 a^4}{L^2}(1-\epsilon^2)$$

$$\Rightarrow T^2 = \frac{4 m^2 \pi^2 a^3}{L^2}\left[\frac{L^2}{GMm^2}\frac{1}{1-\epsilon^2}\right](1-\epsilon^2)$$

$$\Rightarrow T^2 = \frac{4\pi^2}{GM} a^3$$

Lets start applying what we have learnt.

1)



$v_0 = v_1 = \frac{3}{4}\sqrt{2GM/R}$

How far do the particles reach in each case.

For v_0, $|L| = |\vec{r} \times m\vec{v}| = 0$

$\therefore$ it is not an orbit,

We apply energy conservation,

$$-\frac{GMm}{R} + \frac{1}{2} m v_r^2 = -\frac{GMm}{R'}$$

$$\Rightarrow -\frac{GMm}{R} + \frac{1}{2} m \frac{9}{16} \frac{2GM}{R} = -\frac{GMm}{R'}$$

$$\Rightarrow \frac{m}{R}\left(1 - \frac{9}{16}\right) = \frac{m}{R'}$$

$$\Rightarrow R' = \frac{16}{7} R$$

For v_1,

$$|L| = mvR = \frac{3}{4} m \sqrt{2GMR}$$

$$E = \frac{1}{2} m v_0^2 - \frac{GMm}{R}$$

$$= \frac{1}{2} m \frac{9}{16} \frac{2GM}{R} - \frac{GMm}{R}$$

$$= -\frac{GMm}{R}\frac{7}{16}$$

As

$$\frac{1}{r} = \frac{GMm^2}{L^2}\left(1 + \epsilon \cos(\theta)\right)$$

where $\epsilon = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$

$$\frac{1}{r_{max}} = \frac{GMm^2}{\frac{9}{16} m^2\, 2GMR}\left(1 + \frac{2GMm}{R}\frac{7}{16}\frac{9}{16}\frac{m^2\, 2GMR}{G^2M^2m^3}\right)$$

$$\frac{1}{r_{max}} = \frac{8}{9R}\frac{127}{64}$$

$$r_{max} = \frac{72}{127} R$$

2) A satellite of mass m orbits earth with perigee of 1.5Re and apogee of 2.5Re.
i) Find eccentricity
ii) Energy of satellite
iii) Angular Momentum

i) $r_{min} = \frac{d}{1+\epsilon}$

$r_{max} = \frac{d}{1-\epsilon}$

$$\Rightarrow 1.5 R_e + 1.5 R_e \epsilon = d$$
$$2.5 R_e - 2.5 R_e \epsilon = d$$

$$\Rightarrow R_e - 4 R_e \epsilon = 0$$

$$\Rightarrow \boxed{\epsilon = 0.25}$$

$$d = \left(\tfrac{3}{2} + \tfrac{3}{2}\cdot\tfrac{1}{4}\right) R_e$$
$$= \tfrac{15}{8} R_e$$

Now $d = \frac{L^2}{GMm^2}$

$$\therefore L = \sqrt{GMm^2 d}$$

Finally, $\epsilon = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$

$$\frac{1}{16} = 1 + \frac{2E\, GMm^2 d}{G^2M^2m^3}$$

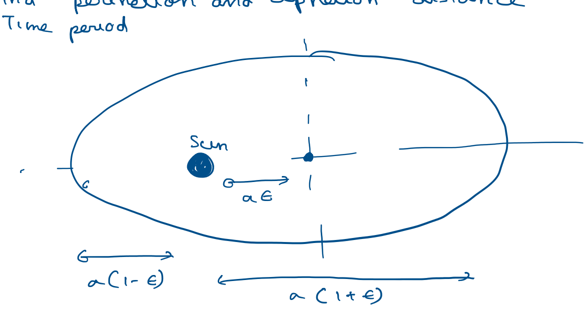
$$\Rightarrow \frac{1}{16} = 1 + \frac{E}{GMm}\frac{15}{4} R_E$$

$$\Rightarrow E\left(\frac{15}{4}\frac{R_E}{GMm}\right) = -\frac{15}{16}$$

$$\Rightarrow E = -\frac{GMm}{4R_E}$$

3) Eccentricity of Halley's comet is 0.967 and its a = 20 AU.
i) Find perihelion and aphelion distance
ii) Time period

i)



$\therefore$ perihelion $= 20\text{ AU}(0.033) = 0.66$ AU
aphelion $= 20\text{ AU}(1.967) = 39.34$ AU

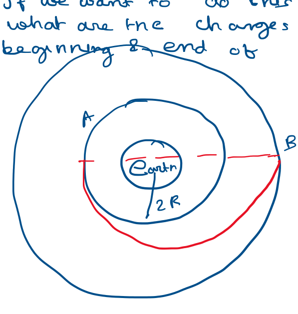
ii)

$$\frac{T^2_{halley}}{T^2_{earth}} = \frac{a^3_{halley}}{a^3_{earth}}$$

$$T = T_{earth}\, 20^{3/2} \text{ years}$$

4) A satellite is on a circular orbit at a radius of 2Re to the earth. We want to transfer it to $4R_E$.

If we want to do this firing the engine only twice, what are the changes in speed required at the beginning & end of motion.



Let v_{ai} be the speed of the satellite before it accelerates & v_{af} is the speed right after it accelerates.
Similarly we define v_{Bi}, v_{Bf} at B

Now, for circular orbit

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\Rightarrow v = \sqrt{GM_e/r} \qquad \therefore v_{Ai} = \sqrt{\frac{GM_e}{2R_e}}$$

$$v_{Bf} = \sqrt{\frac{GM_e}{4R_e}}$$

Further for the transfer orbit, we know that L is constant

$$m\, v_{A,f}\, 2R_E = m\, v_{B,i}\, 4R_E$$

$$\Rightarrow v_{B,i} = v_{A,f}/2$$

Now, using conservation of energy,

$$\frac{1}{2} m v_{A,f}^2 - \frac{GM_e m}{2R_e} = \frac{1}{2} m v_{B,i}^2 - \frac{GM_e m}{4R_e}$$

$$\frac{1}{2} m v_{A,f}^2 \left(\frac{3}{4}\right) = \frac{GM_e m}{4R_E}$$

Solving these,

$$v^2_{A,f} = \frac{2GM_e}{3R_e}$$

$$v^2_{B,i} = \frac{GM_e}{6R_e}$$

$$\Delta v_A = \sqrt{\frac{GM_e}{R_e}}\left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}\right)$$

$$\Delta v_B = \sqrt{\frac{GM_e}{R_e}}\left(\sqrt{\frac{1}{4}} - \sqrt{\frac{1}{6}}\right)$$

5) Qualitative. How would all of our analysis work if $F = -kr^3$?

For the above force,

$$U(r) = -\int_\infty^r \vec{F}(r)\cdot d\vec{r} = br^4/4$$

$$\therefore E = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{L^2}{2mr^2} + \frac{br^4}{4}}_{V_{eff}}$$