Lecture 14

Sunday, 17 November 2019 12:52 AM

We will rush through Cliff's previous lecture and move a class ahead from this point. This will allow us to find space for a final review. Some of you wrote to me that an extra class can be too consuming.

Let us quickly review what Cliff covered in class:

$$\vec{r} = x\hat{i} + y\hat{j}$$

position vector

or $\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

Additionally

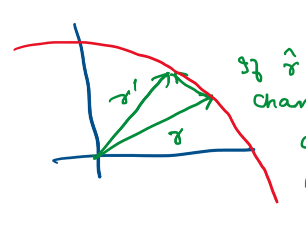
$$\hat{\theta} = \cos(\theta + \pi/2)\,\hat{i} + \sin(\theta + \pi/2)\,\hat{j}$$

$$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Further,

$$d\hat{r}/dr = 0$$

$$d\hat{r}/d\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{\theta}$$



If $\hat{r}$ changes with θ it can be obtained by adding a vector in the direction of $\hat{\theta}$ (tangent)

$$d\hat{\theta}/dr = 0$$

$$d\hat{\theta}/d\theta = -(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = -\hat{r}$$

For completeness

$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \hat{\theta}\,d\theta/dt$$

$$\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = -\hat{r}\frac{d\theta}{dt}$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \underset{V_r}{\dot{r}\hat{r}} + \underset{V_\theta}{r\dot{\theta}\hat{\theta}}$$

$$\vec{L} = \vec{r}\times\vec{p} = m(\vec{r}\times\vec{v})$$

$$= m\vec{r}\times(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$= mr^2\dot{\theta}\,\hat{z}$$

quick review < 10 mins

We will avoid a discussion using this method and use a method we learnt in the previous class.

Gravitational Potential

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

$$\Delta U = -\int_r^\infty \vec{F}\cdot d\vec{r}$$

$$U(\infty) - U(r) = \int_r^\infty \frac{GMm}{r^2}\,dr$$

$$U(r) = U(\infty) - \int_r^\infty \frac{GMm}{r^2}\,dr$$

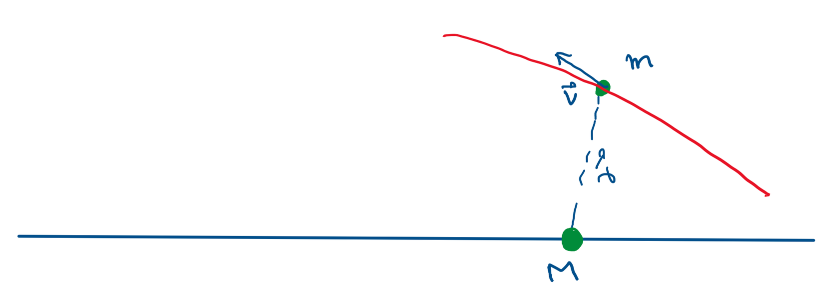
$$= \cancel{U(\infty)} + \frac{GMm}{r}\Big|_r^\infty$$

set to zero

$\Rightarrow U(r) = -GMm/r$ (depends only on r and not $\hat{r}$)

Now lets use this to study a brand new topic of **Orbits**

Let us start with a planet moving under gravity of a star



The only force here is gravity

What are the variables and constants in our situation

$$\frac{dL}{dt} = \vec{r}\times\vec{F}_{grav} = 0$$

$$\frac{dE}{dt} = 0 \quad (\text{No } F_{ext})$$

(Lets further discuss where will the motion be restricted to)
(Use L_0 to argue that it is restricted to the plane of $\vec{r}$ & $\vec{v}$)

Now,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 - \frac{GMm}{r}$$

To analyze the motion we would like to obtain $r(\theta)$

$\therefore$ we rewrite the above in r & θ

$$\Rightarrow E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{GMm}{r}$$

To make our life a little easier we use $L = mr^2\dot{\theta}$

$$\therefore \frac{1}{2}mr^2\dot{\theta}^2 = \frac{L^2}{2mr^2}$$

$$\Rightarrow \frac{m}{2}\left(\frac{dr}{dt}\right)^2 = E - \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

$$\Rightarrow (dr/dt)^2 = \frac{2}{m}\left(E - \frac{L^2}{2mr^2} - \frac{GMm}{r}\right)$$

$$\Rightarrow \frac{(dr/dt)^2}{(d\theta/dt)^2} = \left(\frac{mr^2}{L}\right)^2\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - \frac{GMm}{r}\right) \quad \left[\dot{\theta} = \frac{L}{mr^2}\right]$$

$$\Rightarrow dr/d\theta = \sqrt{\left(\frac{mr^2}{L}\right)^2\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - \frac{GMm}{r}\right)}$$

$\Rightarrow$ Solving for this differential equation, we get

$$\frac{1}{r} = \frac{GMm^2}{L^2}\left(1 + \epsilon\cos(\theta)\right)$$

where $\varepsilon \equiv \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$

L & E are dependent on initial conditions.

1) Find distance of closest approach for this problem

$$\frac{1}{r_{min}} = \frac{GMm^2}{L^2}(1+\epsilon)$$

$$r_{min} = \frac{L^2}{GMm^2}(1+\epsilon)$$

2) How about point of farthest approach

$$r_{max} = \frac{L^2}{GMm^2}\frac{1}{1-\epsilon}$$

$\therefore$ for $\varepsilon \geq 1$ $r_{max} = \infty$

This is because

$$r = \frac{d}{(1+\epsilon\cos\theta)}$$ describe a conic section

$e = 0$ circle

$e < 1$ ellipse

$e = 1$ parabola

$e > 1$ hyperbola

You can see this by checking,

$$r(1+\epsilon\cos\theta) = d$$

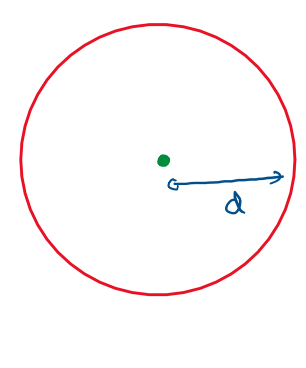
$$r + \epsilon x = d$$

$$\Rightarrow r^2 = (d - \epsilon x)^2$$

$$\Rightarrow x^2 + y^2 = d^2 + \epsilon^2x^2 - 2d\epsilon x$$

for $\epsilon = 0$

$$x^2 + y^2 = d^2$$



$$d = \frac{L^2}{GMm^2}$$

$$d = \frac{m^2v^2d^2}{GMm^2}$$

$$\boxed{\frac{GMm}{d^2} = \frac{mv^2}{d}}$$

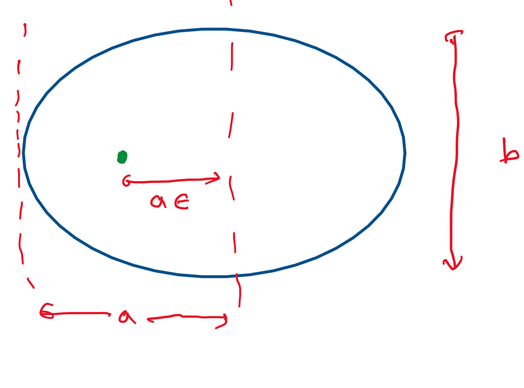
what we have used before

for $0 < \epsilon < 1$

$$\frac{\left(x + \frac{d\epsilon}{1-\epsilon^2}\right)^2}{a^2} + \frac{(y^2)}{b^2} = 1$$

where $a = d/(1-\epsilon^2)$

$$b = d/(1-\epsilon^2)^{1/2}$$

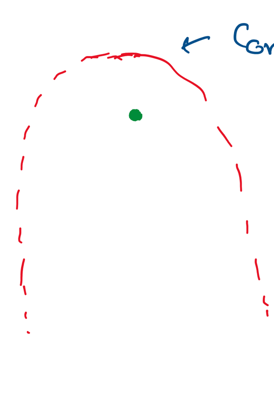


for $\varepsilon = 1$

$$x^2 + y^2 = d^2 + x^2 - 2dx$$

$$y^2 = d^2 - 2dx$$

$$y = -2d(x - d/2)$$

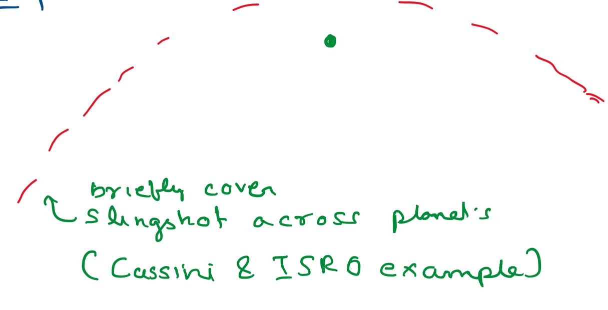


for $\varepsilon > 1$

$$\frac{\left(x - \frac{d\epsilon}{\varepsilon^2-1}\right)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$a = d/(\varepsilon^2 - 1)$$

$$b = d/(\varepsilon^2-1)^{1/2}$$



briefly cover slingshot across planet's (Cassini & ISRO example)

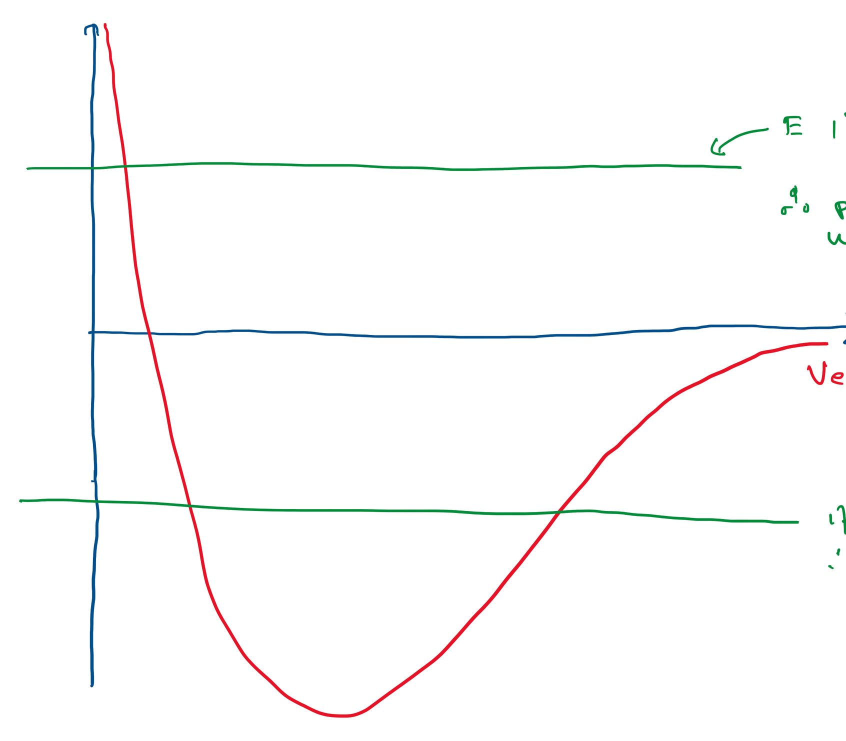
How do we explain all of this using the energy picture?

$$e = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$$

$$\therefore E = \frac{G^2M^2m^3}{2E}(e^2 - 1)$$

Total energy

$$U(eff) = \frac{L}{2mr^2} - \frac{GMm}{r}$$



(I shall not cover transfer orbits as they are currently of not much interest to us)