Lecture 13

Wednesday, 13 November 2019 12:28 AM

Quick discussion on Anal & Prac Track

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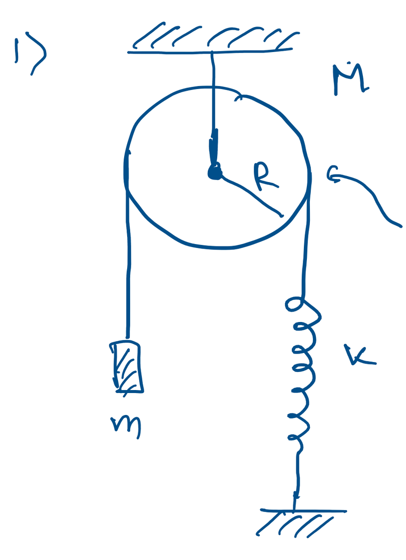
1)



Find ω?

thread doesn't slip

Step 1: Find Equilibrium

$$mgR = KxR \quad \} \text{ balancing torque about pulley}$$

$$x = mg/K$$

Step 2: Write $F = ma$ or $\tau = I\alpha$ with slight displacement

$$\tau = -K(x'+x)R + mgR$$

$$\frac{d}{dt}\left[I\omega + mvR\right] = -Kx'R - \cancel{KxR} + \cancel{mgR}$$

$$I\alpha + mRa = -Kx'R$$

$$I\alpha + mR^2\alpha = -KR^2\theta$$

$$\alpha = -\frac{KR^2}{[I + mR^2]}\theta$$

$$\therefore \omega = \sqrt{\frac{KR^2}{I + mR^2}}$$

In the previous class we studied that SHM occurs under the potential of the form $U = \frac{1}{2}kx^2$.

The generalized $x(t)$ for such a potential is

$$x(t) = A\cos(\omega t + \delta)$$

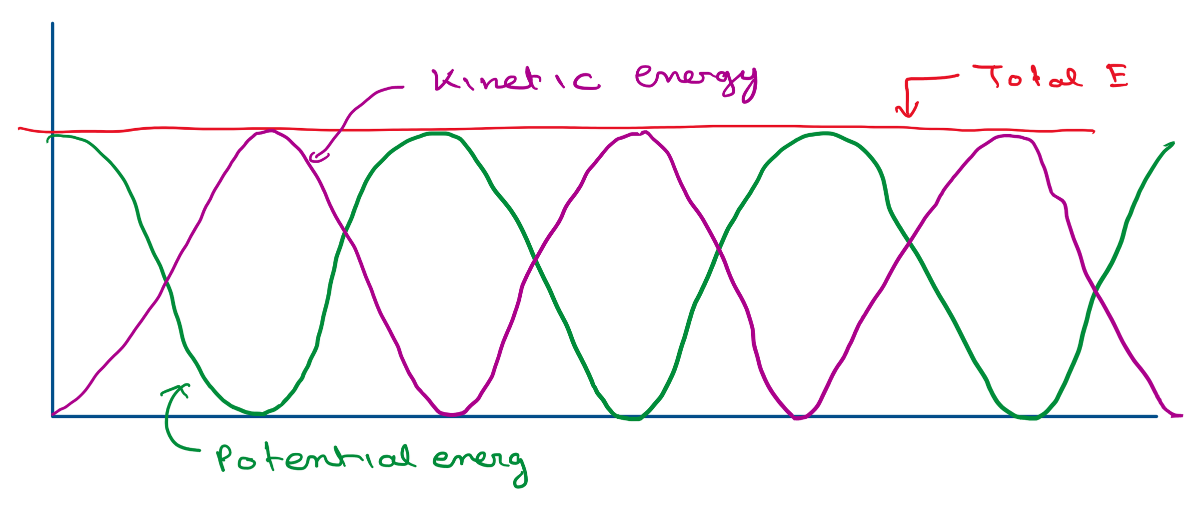
$$\Rightarrow v(t) = -A\omega\sin(\omega t + \delta) \quad \text{where } \omega = \sqrt{k/m}$$

$\therefore$ Energy (mechanical) $= \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

$$= \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \delta) + \frac{1}{2}KA^2\cos^2(\omega t + \delta)$$

$$= \frac{1}{2}KA^2 \leftarrow \text{energy at maximum distance}$$

$\therefore$ energy varies as



To prove something is SHM, you can just show that

$$\boxed{\frac{1}{2}k(A^2 - x^2) = \frac{1}{2}mv^2}$$

For a damped harmonic oscillator where

$$x(t) = \underbrace{\boxed{Ce^{-\beta t/2}}}_{A}\cos(\omega' t + \delta)$$

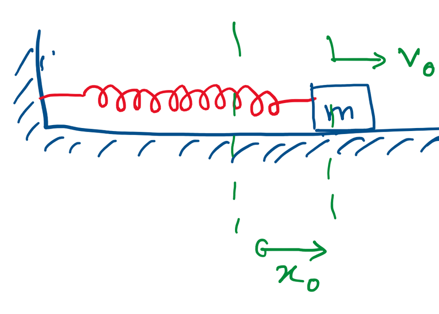
$$\therefore E = \frac{1}{2}kC^2\underbrace{e^{-\beta t}}_{\text{falls exponentially}}$$

Additionally, you can define Q factor for damped oscillators

$$Q \sim \omega_0/\beta$$

$1/Q \sim$ energy lost in one oscillation

2) Consider I release a mass in the following condition



i) Find angular frequency
ii) Find amplitude of oscillations
iii) How long does it take for mass to return to equilibrium

i) $F = -kx \Rightarrow \omega = \sqrt{k/m}$

ii) $\frac{1}{2}kA^2 = \frac{1}{2}kx_0^2 + \frac{1}{2}mv_0^2$

$$\Rightarrow A = \sqrt{x_0^2 + \frac{m}{k}v_0^2}$$

$$\Rightarrow A = \sqrt{x_0^2 + (v_0/\omega)^2}$$

iii) $x(t) = A\cos(\omega t + \delta)$

$v(t) = -A\omega\sin(\omega t + \delta)$

At $t = 0$
$x = x_0$
$v = v_0$

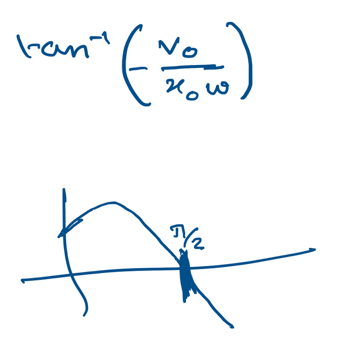
$\therefore$ $x_0 = A\cos\delta$
$v_0 = -A\omega\sin\delta$ $\Rightarrow \tan\delta = -\frac{v_0}{x_0\omega}$

$$\delta = \tan^{-1}\left(-\frac{v_0}{x_0\omega}\right)$$

It returns to equilibrium when

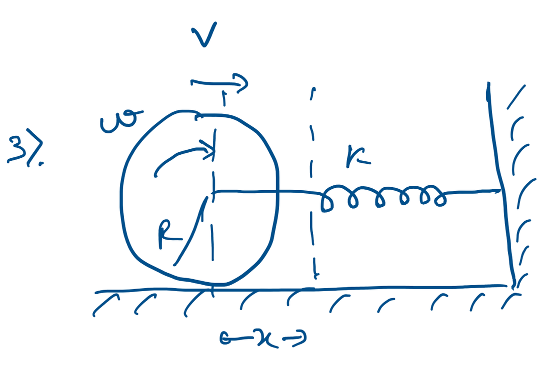
$$x(t) = 0$$

$$\therefore \omega t + \delta = \pi/2$$



$$t = \frac{\pi/2 + \tan^{-1}(v_0/x_0\omega)}{\omega}$$

3)



Mass $= M$ \} cylinder
Radius $= R$
It rolls without slipping

a) Find the energy
b) Find the time period.

a) $E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2$

b) As its rolling without slipping, the friction does no work.
$\therefore$ E is conserved.

$$\frac{dE}{dt} = 0 = Mva + I\omega\alpha + kxv$$

further, $\omega = v/R$

$\alpha = a/R$

$$\Rightarrow 0 = Mva + \frac{I}{R^2}va + kxv$$

$$\Rightarrow 0 = Mva + \frac{M}{2}va + kxv$$

$$\Rightarrow \frac{3}{2}Ma = -kx$$

$$\Rightarrow a = -\frac{2k}{3M}x$$

$$\therefore \omega = \sqrt{2k/3M}$$

$$\therefore T = 2\pi\sqrt{\frac{3M}{2k}}$$

So far we have excluded any external driving force in our SHM.

Such an oscillator would be described by

$$F_{ext} = -kx - r\dot{x} + F_0\sin(\omega t)$$

$$\Rightarrow m\ddot{x} + r\dot{x} + kx = F_0\sin(\omega t)$$

$$\Rightarrow \ddot{x} + \left(\frac{r}{m}\right)\dot{x} + \frac{k}{m}x = \underbrace{\left(F_0/m\right)}_{a_0}\sin(\omega t)$$

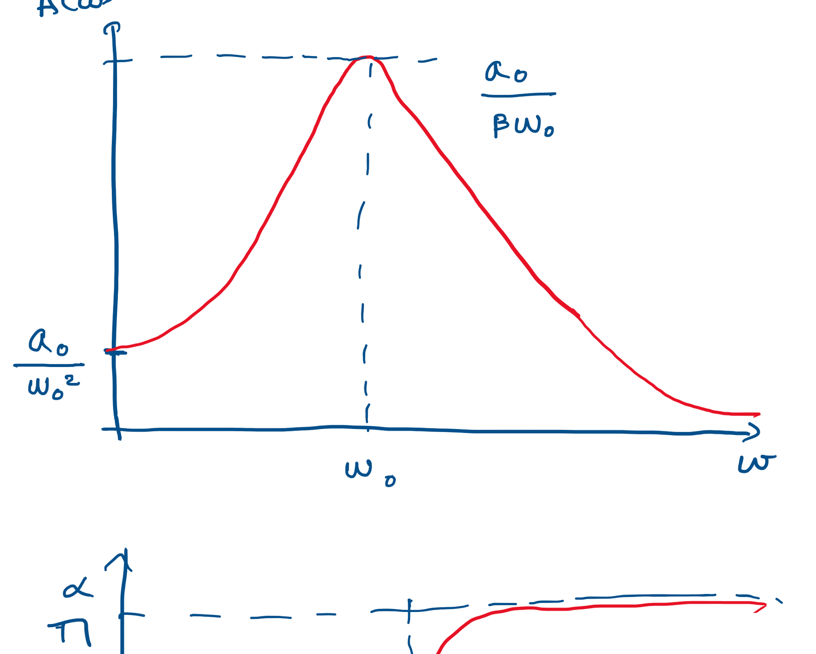
Any equation of this form has a solution:

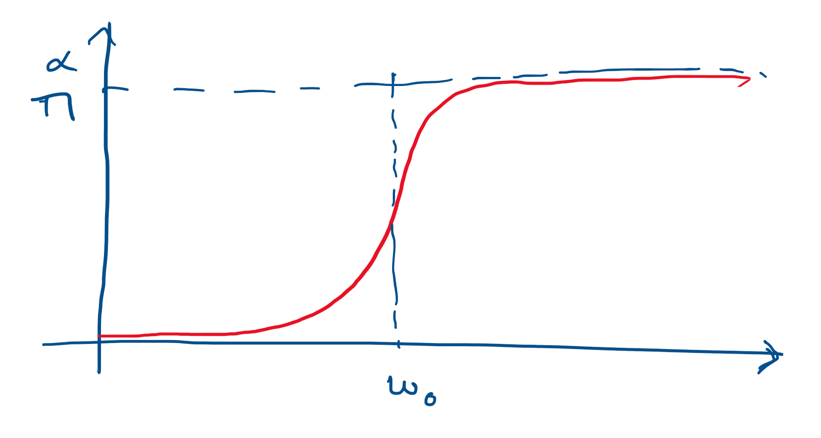
$$X(t) = \underbrace{Ce^{-\beta t/2}\cos(\omega_1 t + \delta)}_{\text{dies out}} + A\sin(\omega t - \alpha)$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2/4}$

$$\approx A\sin(\omega t - \alpha)$$

where $A = \frac{a_0}{\sqrt{(\beta\omega)^2 + (\omega_0^2 - \omega^2)^2}}$ & $\alpha = \tan^{-1}\left(\frac{\beta\omega}{\omega_0^2 - \omega^2}\right)$





(We will avoid solving problems in this area)