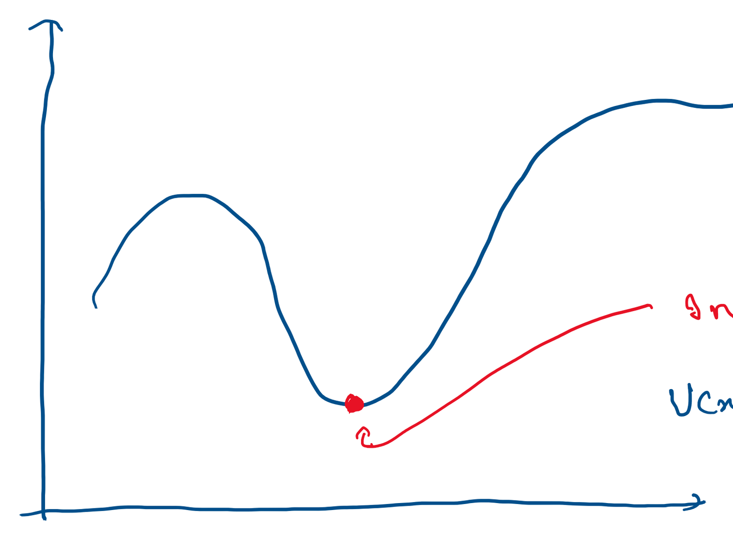
Lecture 12

Sunday, 10 November 2019 11:18 PM

Let us assume any potential function



In this vicinity

just a reference

$$U(x) = \cancel{U(x_0)} + U'(x)(x-x_0) + \frac{U''(x)}{2!}(x-x_0)^2$$

$$\therefore U(x) \approx \frac{U''(x_0)}{2!}(x-x_0)^2$$

$$\therefore U(x) \approx \frac{1}{2}k(x-x_0)^2$$

All equilibriums come with harmonic potentials.

$$U = \frac{1}{2}kx^2$$

$$\text{Now } -\frac{dU}{dx} = -kx$$

$$ma = -kx$$

$$\ddot{x} = -k/m\, x$$

For any equations of this form

$$x(t) = A\cos(\omega t + \delta) \quad \& \quad \omega = \sqrt{k/m}$$

phase shift

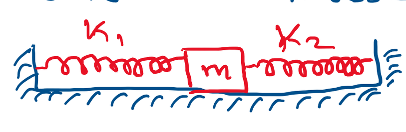
$$\text{or} \quad x(t) = C\cos\omega t + D\sin\omega t$$

$$\text{or} \quad x(t) = E e^{i\omega t} + F e^{-i\omega t}$$

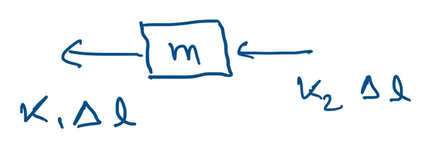
You always need two initial conditions to find out the constants.

Lets apply these to a problem

1) Determine the period of small longitudinal oscillations of a body with mass m and values of spring K_1 and K_2. The friction and the masses of the springs are negligible.



Let the original length be l & change be Δl



$$\therefore \text{Force} = -(K_1+K_2)\Delta l$$

$$\ddot{x} = -\frac{(K_1+K_2)}{m}\Delta l$$

$$\therefore \omega = \sqrt{\frac{K_1+K_2}{m}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K_1+K_2}}$$

2) A particle of mass m moves due to the force $\vec{F} = -\alpha m\vec{r}$ where α is a positive constant, $\vec{r}$ is the radius vector of the particle relative to the origin of coordinates. Find the trajectory of its motion if at initial moment $\vec{r} = r_0\hat{i}$ & $\vec{v} = v_0\hat{j}$.

$$\vec{F} = -\alpha m\vec{r}$$

$$= -\alpha m(x\hat{i} + y\hat{j})$$

For x

$$F_x = -\alpha m x$$

$$a_x = -\alpha x \Rightarrow \omega_1 = \sqrt{\alpha}$$

$$\therefore x = A\sin(\sqrt{\alpha}\, t + \phi)$$

$$\text{at } t = 0,\ x = r_0 \ \& \ v_x = 0$$

$$\therefore r_0 = A\sin\phi$$

$$\& \quad v_x = A\sqrt{\alpha}\cos(\sqrt{\alpha}\, t + \phi)$$

$$0 = A\sqrt{\alpha}\cos(\phi)$$

$$\therefore \phi = \pi/2$$

$$x = r_0\sin(\sqrt{\alpha}\, t + \pi/2)$$

$$= r_0\cos(\sqrt{\alpha}\, t)$$

For y

$$F_y = -\alpha m y$$

$$y = B\sin(\sqrt{\alpha}\, t + \delta)$$

$$\text{at } t = 0,\ y = 0 \ \& \ v = v_0$$

$$0 = B\sin\delta$$

$$\therefore \delta = 0$$

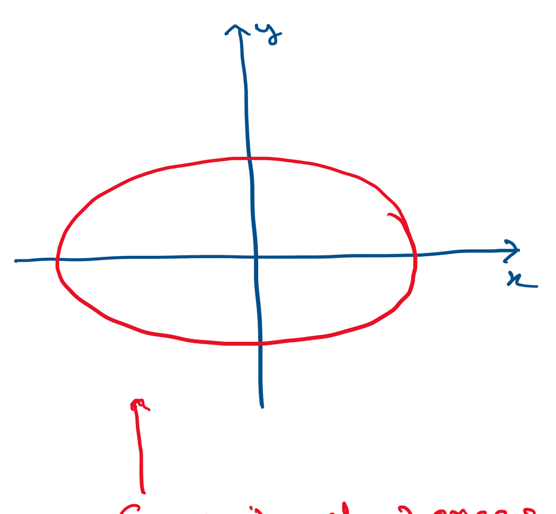
$$v_y = B\sqrt{\alpha}\cos(\sqrt{\alpha}\, t)$$

$$\Rightarrow v_0 = B\sqrt{\alpha}$$

$$\Rightarrow B = v_0/\sqrt{\alpha}$$

$$y = v_0/\sqrt{\alpha}\,\sin(\sqrt{\alpha}\, t)$$

$$\therefore \left(\frac{x}{r_0}\right)^2 + \left(\frac{y}{v_0/\sqrt{\alpha}}\right)^2 = 1$$



Centrifugal forces

Damped Harmonic Oscillator

$$m\ddot{x} = -kx - r\dot{x}$$

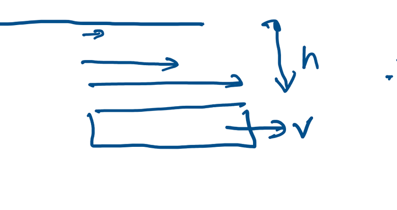
Damping

$$\ddot{x} + \frac{r}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\Rightarrow \ddot{x} + \beta\dot{x} + \omega_0^2 x = 0$$

$$x(t) = Ce^{-\beta t/2}\cos\left(\sqrt{\omega_0^2 - \beta^2/4}\, t + \delta\right)$$

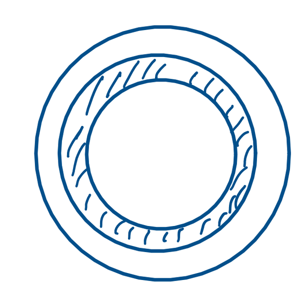
3) A disc of radius R suspended by a rod between two stationary plates and performs torsional oscillation ($\tau_{restoring} = -b\theta$). The clearance of the planes is equal to h, with h << R & viscosity is η. Find the $\theta(t)$ if its released from rest with a twist of θ_0



$$\therefore F_{viscous} = \eta A\, dv/dz$$

$$= \frac{\eta A v}{h}$$

$\therefore$



$$dF = 2\left[\eta\, 2\pi r\, dr\, \frac{\omega r}{h}\right]$$

$$d\tau = -2\left[\eta\, 2\pi r\, dr\, \frac{\omega r}{h}\right] r$$

$$\therefore \tau = -\frac{4\pi\eta\omega}{h}\int r^3\, dr$$

$$= -\frac{\pi\eta\omega}{h}R^4$$

$$\therefore \tau_{net} = -b\theta - \frac{\eta\pi\omega R^4}{h}$$

$$\Rightarrow I\ddot{\theta} + \frac{\eta\pi R^4}{h}\dot{\theta} + b\theta = 0$$

$$\Rightarrow \ddot{\theta} + \underbrace{\boxed{\frac{\eta\pi R^4}{hI}}}_{\beta}\dot{\theta} + \underbrace{\left(\frac{b}{I}\right)}_{\omega_0^2}\theta = 0$$

$$\therefore \theta(t) = Ce^{-\beta t/2}\cos\left(\sqrt{\omega_0^2 - \beta^2/4}\, t + \delta\right)$$

$$\therefore \text{at } t = 0$$

$$\theta_0 = C\cos\delta$$

$$0 = C\sqrt{\omega_0^2 - \beta^2/4}\,\sin\delta$$

$$\therefore \delta = 0 \ \& \ c = \theta_0$$

$$\theta(t) = \theta_0 e^{-\beta t/2}\cos\left(\sqrt{\omega_0^2 - \beta^2/4}\, t\right)$$