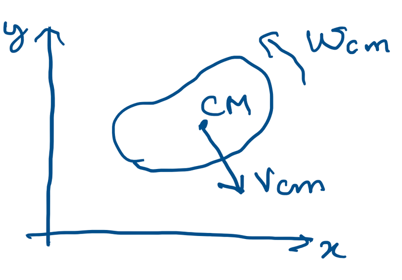
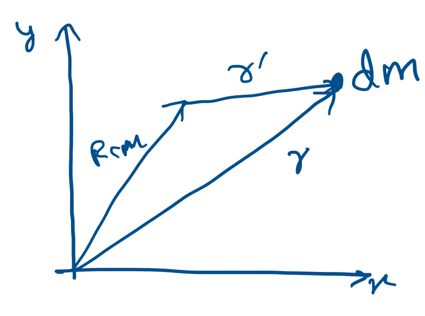
Lecture 10

Thursday, 31 October 2019 2:24 PM

In the last class, we only discussed about bodies hinged about an axis. Lets discuss about freely moving rotating bodies.





$$\therefore \vec{L} = \int (\vec{\gamma} \times \vec{v})\, dm$$

$$= \int (\vec{R}_{cm} + \vec{\gamma}') \times (\vec{V}_{cm} + \vec{v}')\, dm$$

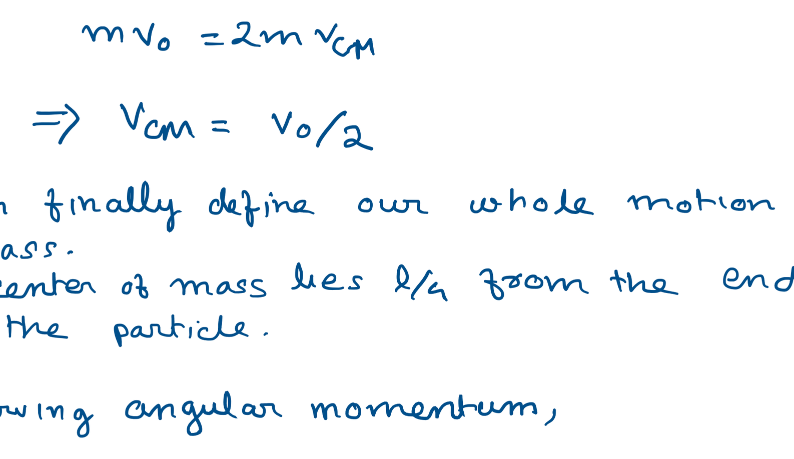
$$= \int (\vec{R}_{cm} \times \vec{V}_{cm})\, dm + \int (\vec{R}_{cm} \times \vec{v}' + \vec{\gamma}' \times \vec{V}_{cm})\, dm + \int \vec{\gamma}' \times \vec{v}'\, dm$$

0 (as γ' & v' are described in the CM frame)

$$= M(\vec{R}_{cm} \times \vec{V}_{cm}) + \int (\vec{\gamma}')^2 \omega_{cm} \hat{z}\, dm$$

$$= M(\vec{R}_{cm} \times \vec{V}_{cm}) + I_{cm} \omega_{cm} \hat{z}$$

1) A mass 'm' travels with speed v_0 perpendicularly to a stick with the same mass 'm' and length 'l' which is initially at rest. The mass collides inelastically with the stick at one of its ends, and sticks to it. What is the resulting angular velocity of the system?



Lets just write down what we readily know

$$m v_0 = 2m v_{CM}$$

$$\Rightarrow V_{cm} = v_0/2$$

We can finally define our whole motion about the center of mass.
This center of mass lies $l/4$ from the end of the stick with the particle.

Conserving angular momentum,

$$m v_0\, l/4 = I_{cm}\, \omega$$

$$\Rightarrow m v_0\, l/4 = \left(\int_{-3l/4}^{l/4} dm\, r^2 + m(l/4)^2 \right) \omega$$

$$\Rightarrow m v_0\, l/4 = \left(\int_{-3l/4}^{l/4} \frac{M}{l} r^2 dr + \frac{m l^2}{16} \right) \omega$$

$$\Rightarrow \cancel{m} v_0\, l/4 = \left(\frac{\cancel{m}}{\cancel{l}} \frac{1}{3} \left(\frac{l^{\cancel{3}\,2}}{64} + \frac{27 l^{\cancel{3}\,2}}{64} \right) + \frac{\cancel{m} l^{2}}{16} \right) \omega$$

$$\Rightarrow v_0 = \left(\frac{1}{3} \left(\frac{l}{16} + \frac{27 l}{16} \right) + \frac{l}{4} \right) \omega$$

$$\Rightarrow v_0 = \left(\frac{28 l}{48} + \frac{l}{4} \right) \omega$$

$$\Rightarrow v_0 = \frac{5}{6} l \omega \quad \Rightarrow \quad \omega = \frac{6 v_0}{5 l}$$

How about the energy of these bodies?

$$KE = \int \frac{1}{2} v^2\, dm$$

$$= \int \frac{1}{2} (\vec{v} \cdot \vec{v})\, dm$$

$$= \int \frac{1}{2} (\vec{V}_{CM} + \vec{v}') \cdot (\vec{V}_{CM} + \vec{v}')\, dm$$

$$= \frac{1}{2} M V_{CM}^2 + \int \vec{V}_{CM} \cdot \vec{v}'\, dm + \int \frac{1}{2} (v')^2\, dm$$

0 in C.M. frame

$$= \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \int (r')^2 \omega_{CM}^2\, dm$$

$$= \frac{1}{2} M V_{CM}^2 + \frac{1}{2} I_{CM} \omega_{CM}^2$$

2) A mass travels with speed v_0 perpendicular to a stick with the mass 'm' and length 'l', which is initially at rest. At what height 'h' above the center of the stick should the mass collide elastically with the stick, so that the mass and center of the stick move with equal speed 'v' after the collision?

Conservation of momentum yields $m v_0 = 2 m v$

$$V = v_0/2$$

Conservation of energy yields $\frac{1}{2} m v_0^2 = \frac{1}{2} m v^2 + \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$

$$\therefore \omega = \left(\frac{m}{2I} \right)^{1/2} v_0$$

Conservation of angular momentum yields

$$m h v_0 = m h v + I \omega$$

$$\therefore h = (2I/m)^{1/2} = \frac{l}{\sqrt{6}}$$

Let us reanalyze some of the above results in specific scenarios. If the ω of the CM about the origin is the same as the 'ω' measured at the center of mass.

$$L_z = M(\vec{R}_{cm} \times \vec{V}_{cm}) + I_{cm} \vec{\omega}_{cm}$$

$$= M \omega R_{CM}^2 + I_{cm} \omega$$

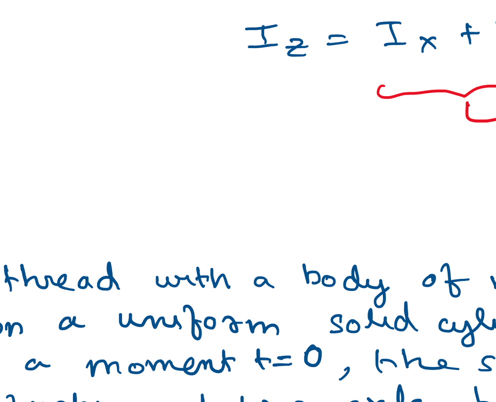
$$= (M R_{CM}^2 + I_{CM}) \omega = I' \omega$$

$$\therefore I \text{ about Origin} = I_{CM} + M R_{CM}^2$$

This holds for rigid bodies

This is known as the parallel axis theorem.

Further,



$$I_z = \int r^2\, dm = \int (x^2 + y^2)\, dm$$

Similarly

$$I_x = \int (y^2 + z^2)\, dm$$

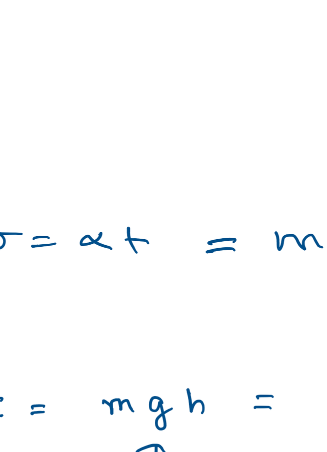
$$I_y = \int (x^2 + z^2)\, dm$$

For a body with no height in the z direction, i.e. $z = 0$

$$I_z = I_x + I_y$$

Valid for wafers only

3) A light thread with a body of mass 'm' tied to its end is wound on a uniform solid cylinder of mass M and radius 'R'. At a moment t=0, the system is set in motion. Assume friction at the axle to be negligible. Find
a) angular velocity
b) KE of the whole system



i) $mg - T = ma$

ii) $TR = \frac{MR^2}{2} \alpha = \frac{MRa}{2}$

$$\Rightarrow T = \frac{Ma}{2}$$

$$\therefore a = \frac{mg}{(m + M/2)}$$

$$\therefore \alpha = \frac{mg}{(m + M/2) R}$$

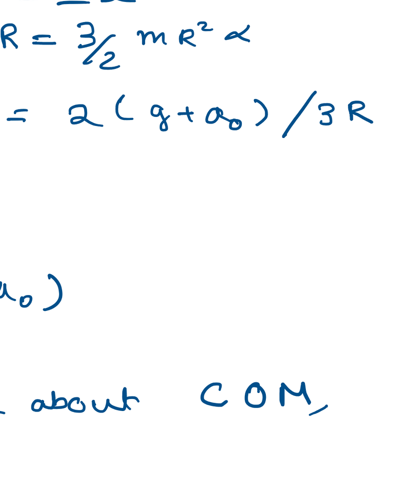
$$\omega = \alpha t = mgt / (m + M/2) R$$

$$KE = mgh = \frac{1}{2} \frac{m^2 g^2 t^2}{(m + M/2)}$$

(mgh: Change in potential energy)

4) Thin threads are tightly wound on the ends of a uniform solid cylinder of mass 'm'. The free ends of the thread are tied to the cieling of a elevator. The elevator starts moving up with an acceleration of 'a_0'. Find the acceleration 'a'' of the cylinder relative to the elevator and the force exerted by the cylinder on the ceiling.

In the frame of the lift,



$$\tau = I \alpha$$

$$m(g + a_0) R = \frac{3}{2} m R^2 \alpha$$

$$\Rightarrow \alpha = 2(g + a_0)/3R$$

$$\therefore a' = \alpha R = \frac{2}{3}(g + a_0)$$

Now, doing the same about COM,

$$TR = I \alpha$$

$$\Rightarrow T = \frac{mR^2}{2} \frac{2}{3R^2} (g + a_0)$$

$$\Rightarrow T = (g + a_0)/3$$