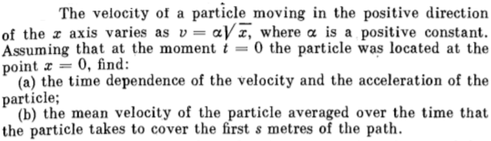
PH1A – FINAL REVIEW

Saturday, December 7, 2019 1:52 AM

QUESTION 1:



The velocity of a particle moving in the positive direction of the x axis varies as $v = \alpha\sqrt{x}$, where α is a positive constant. Assuming that at the moment $t = 0$ the particle was located at the point $x = 0$, find:

(a) the time dependence of the velocity and the acceleration of the particle;

(b) the mean velocity of the particle averaged over the time that the particle takes to cover the first s metres of the path.

SOLUTION 1:

a) $v = \alpha\sqrt{x} = \alpha x^{1/2}$

$$\therefore a = \frac{dv}{dt} = \frac{\alpha}{2}\frac{1}{x^{1/2}}\frac{dx}{dt}$$

$$= \frac{\alpha v}{2x^{1/2}}$$

$$= \frac{\alpha(\alpha x^{1/2})}{2x^{1/2}} = \frac{\alpha^2}{2}. \quad //$$

Since acceleration is constant,

$v = u + at$ where u is the velocity at $t=0$.

At $t=0$, $x=0$, $\therefore v = \alpha\sqrt{x} = 0$

$$\therefore v = at = \frac{1}{2}\alpha^2 t. \quad //$$

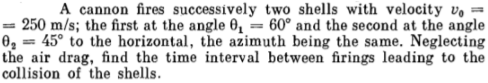
b) $s = ut + \frac{1}{2}at^2 = \frac{1}{4}\alpha^2 t^2$

Therefore, the time taken to cover s meters is

$$t = \frac{2\sqrt{s}}{\alpha}.$$

$$\text{Mean velocity} = \frac{s}{t} = \frac{s\alpha}{2\sqrt{s}} = \frac{\sqrt{s}\,\alpha}{2}. \quad //$$

QUESTION 2:



A cannon fires successively two shells with velocity $v_0 = 250$ m/s; the first at the angle $\theta_1 = 60°$ and the second at the angle $\theta_2 = 45°$ to the horizontal, the azimuth being the same. Neglecting the air drag, find the time interval between firings leading to the collision of the shells.

SOLUTION 2 :

Let the time difference between firing the two shells be Δt.

For the first shell :

$$x_1 = v_0 \cos\theta_1 (t+\Delta t)$$

$$y_1 = V_0 \sin\theta_1 (t+\Delta t) - \frac{1}{2} g (t+\Delta t)^2$$

and for the second shell :

$$x_2 = V_0 \cos\theta_2 t$$

$$y_2 = V_0 \sin\theta_2 t - \frac{1}{2} g t^2 .$$

For the two to collide,

$$x_1 = x_2 \quad \text{and} \quad y_1 = y_2 .$$

$$\Rightarrow \quad V_0 \cos\theta_1 (t+\Delta t) = V_0 \cos\theta_2 t \quad -(1)$$

and $$V_0 \sin\theta_1 (t+\Delta t) - \frac{1}{2} g\left(t^2 + (\Delta t)^2 + 2t\Delta t\right) = V_0 \sin\theta_2 t - \frac{1}{2} g t^2 . \quad -(2)$$

From (1),

$$t = \frac{\cos\theta_1 \Delta t}{\cos\theta_2 - \cos\theta_1} .$$

Substituting this expression for t in (2) gives,

$$V_0 \sin\theta_1 \Delta t \left(\frac{\cos\theta_2}{\cos\theta_2 - \cos\theta_1}\right) - \frac{g (\Delta t)^2}{2} \left(\frac{\cos\theta_1 + \cos\theta_2}{\cos\theta_2 - \cos\theta_1}\right)$$

$$= \frac{V_0 \sin\theta_2 \cos\theta_1 \Delta t}{\cos\theta_2 - \cos\theta_1}$$

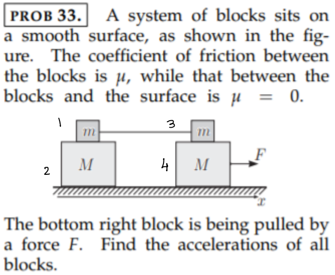
The two solutions are : $\Delta t = 0$,

and $$\Delta t = \frac{2V_0}{g} \frac{(\sin(\theta_1 - \theta_2))}{(\cos\theta_1 + \cos\theta_2)} .$$

$\Delta t = 0$ corresponds to both being fired at the same time.

Plugging in the values, $\Delta t \sim 11\,s$.

QUESTION 3:


1
3
m
m
2
M
4
M
F
x

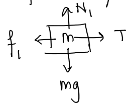
PROB 33. A system of blocks sits on a smooth surface, as shown in the figure. The coefficient of friction between the blocks is μ, while that between the blocks and the surface is $\mu = 0$.

The bottom right block is being pulled by a force F. Find the accelerations of all blocks.

SOLUTION :

The free body diagrams for the blocks are :

1:



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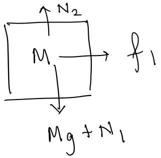

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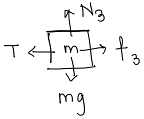
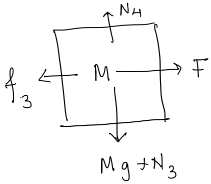


4:



When bodies are connected by frictional forces, all possible combinations of there being relative slipping between all the touching surfaces need to be considered.

Case A :

If none of the upper blocks are slipping over the lower blocks, we can consider the system as a whole, moving at an acceleration a.

In this case, $a_1 = a_2 = a_3 = a_4 = a = \frac{F}{2(M+m)}$,

In order for this to be possible,

$f_3 \leqslant \mu mg$.

f_3 accelerates the system composed of blocks 1, 2 and 3.

$\therefore \quad f_3 = (2m+M)\,a$.

$$= \frac{(2m+M)F}{2(m+M)}$$

$\therefore$ For case A, $F \leqslant \frac{2\mu mg(m+M)}{(2m+M)}$.

CASE B:

If $F > \frac{2\mu mg(m+M)}{(2m+M)}$, blocks 3 and 4 will slip against each other since f_3 has to be less than μmg.

In this case, we have

$F - f_3 = M a_4$

and $f_3 = (M+2m)\,a$ (where a is the acceleration of blocks 1, 2 and 3).

$f_3 = \mu mg$.

$$\therefore \quad a = \frac{\mu m g}{(M + 2m)} \quad \text{and} \quad a_4 = \frac{F - \mu m g}{M}.$$

Note: Block $\underline{1}$ will never slip against Block 2.

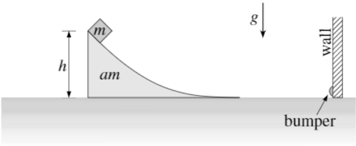
When $f_3 \leq \mu m g$, block 3 moves ahead with

$m a_3 = f_3 - T$ and $T < \mu m g$

For block 1 to slip, $T > \mu m g$ so it will not slip. This is Case A.

For case B, $f_3 = \mu m g$, $T < \mu m g$ for block to accelerate. Therefore, block 1 cannot slip.

QUESTION 4 :



A block of mass m is released and slides without friction down an initially stationary ramp of mass am (where a is some number). The block starts at a height h above the horizontal surface on which the ramp rests, and the ramp itself is free to slide on this surface without friction. The ramp is curved such that the block can slide smoothly off the ramp and onto the horizontal surface. Assume that the physical size of the block is negligible compared to h and compared to the curvature of the ramp.

(a) (1 pt) Consider the system just after the block and ramp separate. Is the total energy the same as it was in the initial configuration shown in the diagram? Is the total linear momentum the same?

(b) (3 pts) Find the velocity of the block (v_b) and the velocity of the ramp (v_r) after the two separate, in terms of some or all of g, h, a and m.

(c) (2 pts) The block, now sliding freely to the right along the horizontal surface, bounces elastically off an immovable bumper, thereby changing its direction but maintaining its speed. For what values of a will the block eventually catch up to the ramp?

(d) (2 pts) Now take $a = 4$. The block catches up to, and begins sliding up on, the ramp. At the moment that the block comes to rest (instantaneously) relative to the ramp, what is the velocity v_f of the ramp/block system?

(e) (2 pts) How high up the ramp does the block reach, expressed as a fraction of the initial height h?

SOLUTION :

(a) (1 pt)
Consider the system comprising the block and the ramp. As the two objects slide apart, the only external forces acting on the system are gravity and the normal force of the table on the ramp. Both are directed vertically, along the $\hat{y}$ direction; no external forces act in the $\hat{x}$ direction (horizontally). The normal force on the ramp does no work. Gravity is a conservative force. Therefore the total energy of the system (the sum of kinetic energy and gravitational potential energy for *both* objects) is conserved. The x-component of linear momentum is also conserved. (Note that the y-component of the total momentum is zero at $t = 0$, as well as after the objects separate, but is nonzero as the block slides down the ramp!)

(b) (3 pts)
The total energy of the system is conserved. Therefore, the total energy after the objects separate must be equal to the total energy at $t = 0$:

$$mgh = \frac{1}{2}mv_b^2 + \frac{1}{2}(am)v_r^2.$$

We can also write down an equation expressing conservation of linear momentum (more precisely, its x-component):

$$0 = mv_b + amv_r \qquad \Longrightarrow v_b = -av_r.$$

Here the positive direction $+\hat{x}$ points to the right, so we will have $v_b \geq 0$. Substitute the second equation into the first:

$$2gh = (-av_r)^2 + av_r^2 = a(a+1)v_r^2.$$

Therefore, we obtain

$$v_r = -\sqrt{\frac{2gh}{a(a+1)}},$$

since the negative branch of the square root is the physically relevant one. It follows from our second equation that

$$v_b = +a\sqrt{\frac{2gh}{a(a+1)}} = +\sqrt{\frac{2agh}{a+1}}.$$

(c) (2 pts)
Since the block bounces elastically off of the wall, $v_b' = -v_b$. The block will catch up to the ramp when $|v_b'| = |v_b| > |v_r|$. But this just means that we must have $|av_r| = a|v_r| > |v_r|$, or $a > 1$.

(d) (2 pts)
Just as in part (a), total energy and horizontal linear momentum are conserved. The block will catch up to the ramp and slide up it for some distance, eventually coming instantaneously to rest at some maximum height h'. At the instant when the two objects are at rest with respect to one another, they will move with the same velocity v_f in the lab frame. Write down conservation of momentum:

$$mv_b' + (am)v_r = (m + am)v_f.$$

Collecting terms and canceling the factor of m, we obtain

$$2av_r = (1+a)v_f. \qquad \Longrightarrow v_f = \frac{2a}{1+a}v_r = -\frac{2a}{1+a}\sqrt{\frac{2gh}{a(a+1)}}.$$

Note that we could have arrived at this equation by identifying $-2mv_b$ as the impulse received by the block in its collision with the wall, which must be the total momentum of the block-ramp system after the block collides with the wall. Setting $a = 4$, this becomes

$$v_f = \frac{2}{5}\sqrt{\frac{8gh}{5}} = \frac{4}{5}\sqrt{\frac{2gh}{5}}.$$

This is just a totally inelastic collision, except that the lost kinetic energy is not "dissipated" into heat, but converted into potential energy as the block slides up on the ramp. Notice also that it is not possible to work this problem straightforwardly in a frame in which the ramp is stationary, because such a frame is not inertial. The velocity of the ramp is not constant during the collision. While there are no *external* horizontal forces on the system, the block exerts a horizontal force on the ramp and it speeds up. Indeed, we found $v_f \neq v_r$.

(e) (2 pts)
We write down conservation of energy:

$$\begin{aligned}\frac{1}{2}(m+am)v_f^2 + mgh' &= \frac{1}{2}mv_b'^2 + \frac{1}{2}amv_r^2\\ &= \frac{m}{2}\left(\frac{2agh}{(a+1)} + \frac{a\cdot 2gh}{a(a+1)}\right)\\ &= mgh.\end{aligned}$$

We could have jumped directly to the third line by noticing that the collision of the block with the wall was perfectly elastic, so that the system's energy is conserved not just for the two "collision" interactions independently but in fact throughout the entire problem.

Setting $a = 4$ and using our answer for part (d),

$$\begin{aligned}mgh &= mgh' + \frac{5}{2}m\frac{2gh}{5}\left(\frac{4}{5}\right)^2\\ &= mgh' + \frac{16}{25}mgh.\end{aligned}$$

Solving for h' gives our answer,

$$h' = \frac{9}{25}h.$$

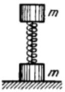
As a common-sense check, we have $h' < h$.

QUESTION 5 :

A system consists of two identical cubes, each of mass m, linked together by the compressed weightless spring of stiffness k. The cubes are also connected by a thread which is burned through at a certain moment. Find:

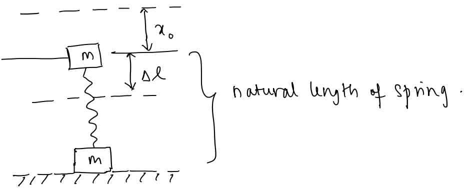
(a) at what values of Δl, the initial compression of the spring, the lower cube will bounce up after the thread has been burned through;

(b) to what height h the centre of gravity of this system will rise if the initial compression of the spring $\Delta l = 7\ mg/k$.



SOLUTION :

(a)



Once the thread is burnt through, the upper block will begin to rise up. The lower block will bounce (rise) when the force on it is greater than mg.

Consider the stretch x_0 of the spring where

$$kx_0 = mg.$$

At this point, by energy conservation,

$$\frac{1}{2}kx_0^2 + \frac{1}{2}mv^2 + mg(x_0+\Delta l) = \frac{1}{2}k(\Delta l)^2$$

where v is the velocity of the upper block and the lower block is at rest. (We can take potential energy of the upper block to be zero initially).

The minimum value of Δl is when $v=0$.

Using $x_0 = mg/k$,

We get : $\frac{1}{2}k(\Delta l)^2 - mg\Delta l - \frac{3m^2g^2}{2k} = 0$

$\therefore \Delta l = \frac{3mg}{k}$. //

(b) If $\Delta l = 7mg/k$,

$$\frac{k}{2}\left(\frac{7mg}{k}\right)^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}k\left(\frac{mg}{k}\right)^2 + mg\left(\frac{mg}{k}+\frac{7mg}{k}\right)$$

where v_0 is the velocity of the upper block when system just leaves the ground.

$\therefore v_0 = \sqrt{\frac{32mg^2}{k}}$.

The initial velocity of the center of mass will be

$$v_{CM}^{i} = \frac{mv_0 + m(0)}{2m} = \frac{v_0}{2} .$$

When the lower block is just about to move, the upper block has risen $\left(\Delta l + \frac{mg}{k}\right)$ and the center of mass has risen $\frac{1}{2}\left(\Delta l + \frac{mg}{k}\right)$.

At its peak height, the center of mass will have zero velocity. By energy conservation for the center of mass,

$$\frac{1}{2}(2m)\left(\frac{v_0^2}{4}\right) + 2mg\left[\frac{1}{2}\left(\Delta l + \frac{mg}{k}\right)\right] = 2mgh$$

where h is the peak height.

$\therefore h = \frac{4mg}{k} + \frac{4mg}{k} = \frac{8mg}{k}$. //

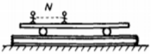
QUESTION 6 :

A railroad flatcar of mass *M* can roll without friction along a straight horizontal track as shown in Fig. 1.86. N men, each of mass m, are initially standing on the car which is at rest.

(a) The N men run to one end of the car in unison; their speed relative to the car is V_r just before they jump off (all at the same time). Calculate the velocity of the car after the men have jumped off.

(b) The N men run off the car, one after the other (only one man running at a time), each reaching a speed V_r relative to the car just before jumping off. Find an expression for the final velocity of the car.

(c) In which case, (a) or (b), does the car attain the greater velocity?



SOLUTION :

(a) As there is no horizontal external force acting, the center of mass of the system consisting of the flatcar and N men remains stationary. Taking the x-axis along the track, we have for the center of mass,

$$x_{\text{cm}} = \frac{Mx_{\text{car}} + Nmx_{\text{man}}}{M + Nm} ,$$

$$\dot{x}_{\text{cm}} = 0 = M\dot{x}_{\text{car}} + Nm\dot{x}_{\text{man}} ,$$

where $\dot{x}_{\text{car}}$ and $\dot{x}_{\text{man}}$ are respectively the velocities of the car and each man after the men have jumped off. Writing $\dot{x}_{\text{car}} = V_{\text{car}}$ and noting that $\dot{x}_{\text{man}} = V_{\text{car}} - V_r$, we have

$$MV_{\text{car}} + Nm(V_{\text{car}} - V_r) = 0 ,$$

giving

$$V_{\text{car}} = \frac{NmV_r}{M + Nm} .$$

(b) Consider the transition from n men to $(n-1)$ men on the car. Let V_n be the velocity of the car when n men are left on it. The total momentum of the car with the n men is

$$P_n = MV_n + nmV_n .$$

When the nth man jumps off the car with a speed V_r relative to the car, the momentum of the system consisting of the car and n men is

$$P_{n-1} = MV_{n-1} + (n-1)mV_{n-1} + m(V_{n-1} - V_r) .$$

Momentum conservation $P_{n-1} = P_n$ gives

$$(M + nm)V_n = (M + nm)V_{n-1} - mV_r ,$$

or

$$V_{n-1} = V_n + \frac{mV_r}{M + nm} .$$

Hence

$$V_{n-s} = V_n + \sum_{i=1}^{s} \frac{mV_r}{M + (n - i + 1)m} .$$

As $n = N$, $V_N = 0$ initially, we have for $s = N$,

$$V_0 = \sum_{i=1}^{N} \frac{mV_r}{M + (N - i + 1)m} = \sum_{n=1}^{N} \frac{mV_r}{M + nm} .$$

(c) As

$$\sum_{n=1}^{N} \frac{1}{M + nm} > \frac{N}{M + Nm} ,$$

the car in case (b) attains a greater final velocity.

QUESTION 7 :

Figure 1

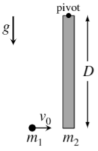


Figure 2

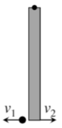
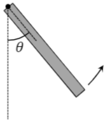


Figure 3



A thin, uniform rod of mass m_2 and length D hangs from a frictionless pivot. A ball of mass m_1 moving horizontally with speed v_0 strikes the lower end of the rod (Figure 1). The collision is elastic. The acceleration due to gravity is g.

(a) (3 pts) Write down the equations for energy and angular momentum that relate the speed of the ball v_0 immediately before the collision to the speed of the ball v_1 and the speed of the end of the rod v_2 immediately after the collision (Figure 2).

(b) (2 pts) Find v_1 and v_2 in terms of some or all of v_0, m_1, m_2, and D.

(c) (2 pts) Find the relation between m_1 and m_2 that must hold if the ball is to reverse its direction in the collision, as depicted in Figures 1 and 2 (*i.e.*, the ball enters the collision moving to the right but leaves the collision moving to the left).

(d) (3 pts) Find the maximum angle θ_{max} of the rod's subsequent motion (Figure 3) in terms of some or all of v_0, m_1, m_2, D, and g.

SOLUTION :

(a.) The ball hits the pendulum at say, $t = 0$. Take the direction to the right to be $+ve$ by convention. For sake of concreteness, assume that the speed of the ball immediately after collision v_1 is positive.
In this collision, total linear momentum of the ball+pendulum system will not be conserved since the pivot will provide a contact force to prevent translation of the pendulum as whole. However, the pendulum starts to rotate about the pivot with some angular speed ω. Angular momentum about the pivot P will be conserved before and after the collision, since there is no net torque acting about P. Note that, even though there is a contact force at P, it does not contribute to the torque since its moment($\vec{r} \times \vec{F}$) about P vanishes. Thus, angular momentum conservation equation about P before and immediately after the collision is.

$$m_1 v_0 D = m_1 v_1 D + I\omega\,, \tag{1}$$

where I is the moment of inertia of the rod about the pivot. Since the rod is of uniform density, and we have $I = \eta m_2 D^2$, with $\eta = 1/3$.
Now, since the collision is elastic, total mechanical energy is a constant of motion too. The energy conservation equation before and just after the collision is therefore,

$$\frac{1}{2}m_1 v_0^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}I\omega^2\,, \tag{2}$$

where we have used the fact that just before and immediately after the collision, the gravitational potential energy of neither the ball nor the pendulum changes, and hence we do not keep track of those terms in order to have equations for v_1 and v_2. Since the pendulum performs rotation about P, the speed of its lowermost point v_2 immediately after collision is related to ω via $v_2 = \omega D$.

(b.) We have two unknowns v_1 and v_2 immediately after the collision and also two equations as determined above. From the angular momentum conservation of Eq. (1), we get,

$$v_1 = v_0 - \eta\frac{m_2}{m_1}v_2\,, \tag{3}$$

which we substitute in Eq. (2) to solve for v_2 to get a quadratic equation,

$$\left(1 + \eta\frac{m_2}{m_1}\right)v_2^2 - \left(\frac{2\eta v_0 m_2}{m_1}\right)v_2 = 0\,. \tag{4}$$

Assuming finite masses for m_1 and m_2, one solution is $v_2 = 0$ which is unphysical in this setup, since we are told that the ball hits the pendulum and must change its speed from 0. The other (correct) solution is,

$$v_2 = \frac{2v_0}{1 + \eta\frac{m_2}{m_1}}\,, \tag{5}$$

and substituting this back in Eq. (3), we obtain,

$$v_1 = \frac{1 - \eta\frac{m_2}{m_1}}{1 + \eta\frac{m_2}{m_1}}v_0\,. \tag{6}$$

(c.) For the ball to rebound its direction in the collision, we demand $v_1 < 0$, since by assumption we assumed it to be positive in solving for Eq. (6). Thus, the rebound condition is(remember, $\eta = 1/3$),

$$1 - \eta\frac{m_2}{m_1} < 0 \implies \frac{m_2}{m_1} > 3\,. \tag{7}$$

(d.) Now we will find the maximum angle swept by the pendulum by using conservation of energy, since gravity is a conservative force and hence, total mechanical energy is conserved. One could, in principle, take the ball into account too while setting up the equations, but then we will have to solve for its subsequent motion under gravity too, which is of course, not hard, but unnecessary in this case. We can just focus on the pendulum. Just after the collision, the total mechanical energy(superscript "p" for pendulum) of the pendulum is only kinetic (taking this configuration as the zero of the gravitational potential energy of the pendulum),

$$E_i^p = \frac{1}{2} I\omega^2 = \frac{1}{2}\eta m_2 v_2^2 \,, \tag{8}$$

where v_2 is given by Eq. (5). At $\theta = \theta_{max}$, the energy is altogether potential in nature since the pendulum momentarily comes to rest. The center of mass of the pendulum is raised by a height $h = \frac{D}{2}\left[1 - \cos(\theta_{max})\right]$ and hence,

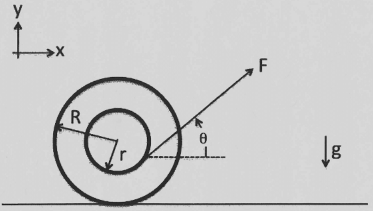
$$E_f^p = m_2 g h \,. \tag{9}$$

Now since mechanical energy of the pendulum will be conserved, we have $E_i^p = E_f^p$, which solves to give,

$$\theta_{max} = \cos^{-1}\left[1 - \frac{4\eta v_0^2}{gD\left(1 + \eta\frac{m_2}{m_1}\right)^2}\right] \,. \tag{10}$$

And of course, $\eta = 1/3$.

QUESTION 8 :



A yo-yo with outer radius R is pulled along a horizontal surface by a force F applied to a string wrapped around a hub of radius r, as shown in the figure. The string lies in the xy plane and makes an angle θ with respect to the horizontal. The mass of the yo-yo is M, and its moment inertia with respect to its center is $I = bMR^2$, where $0 \le b \le 1$. The coefficient of friction between the yo-yo and the surface is μ.

(1.5 points) (a) Find the normal force N exerted by the ground on the yo-yo in terms of M, g, F, and θ.

(2 points) (b) For a particular angle θ, the yo-yo is stationary despite the force F. Find this angle in terms of r and R.

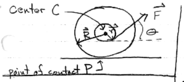
(1.5 points) (c) What is the maximum value of F for which the yo-yo is not lifted from the table?

(2 points) (d) Assuming the yo-yo rolls without slipping, find an expression for the linear acceleration a_x as a function of the variables given in the problem statement and figure.

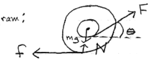
(2 points) (e) In terms of the given quantities, what is the minimum value of the coefficient of friction μ required for the yo-yo not to slip?

(1 points) (f) What direction does the yo-yo roll if $\theta = 0°$? Explain.

SOLUTION :



Free body diagram;



$$\begin{aligned}
\Sigma F_x &= F\cos\theta - f = Ma_x \\
\Sigma F_y &= F\sin\theta + N - Mg = 0 \\
\Sigma \tau_{\text{about } C} &= fR - Fr = I_{\text{about } C}\,\alpha \\
\text{rolling} \quad & a_x = R\alpha
\end{aligned}$$

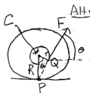
Common Mistakes:

1) $f \neq \mu N$ except when about to slip as in part (e)
2) angle between $\vec{r}$ and $\vec{F}$ is 90° not θ, so torque $|\vec{r} \times \vec{F}| = rF \neq rF\sin\theta$
3) signs; $\tau = +fR$ (clockwise) $- Fr$ (counter clockwise)

a) From ΣF_y, $\boxed{N = Mg - F\sin\theta}$

c) liftoff when $N = 0$, ie when $\boxed{F = Mg/\sin\theta}$

b) stationary: $\left\{\begin{array}{l} a_x = 0 \rightarrow f = F\cos\theta \\ \alpha = 0 \rightarrow f = Fr/R \end{array}\right\} \rightarrow \boxed{\cos\theta = \frac{r}{R}}$



Alternative proof: consider torque about point of contact P. All forces except F always act through P and thus never exert torques about that point. At the particular angle θ where $\vec{F}$ also acts along a line through P, there is no torque at all about P so the yoyo does not roll. For this θ, $\cos\theta = r/R$ (see triangle connecting C, P, and Q).

d) In box at top of page, combining $\Sigma\tau$ and rolling equations → $\boxed{f - \frac{Fr}{R} = bMa_x}$

Add this relation to ΣF_x equation at top to obtain

$$\boxed{a_x = \frac{F}{M}\,\frac{(\cos\theta - r/R)}{(1+b)}}$$

Note that $a_x = 0$ at $\cos\theta = r/R$ as we found in part (b).

e) Yoyo starts to slip when f reaches its maximum

$$\boxed{f = \mu N = \mu(mg - F\sin\theta)} \quad (\text{using } \Sigma F_y)$$

Eliminating a_x from the two boxed relations in (d), we also have

$$\boxed{f = \frac{F}{1+b}\left[\frac{r}{R} + b\cos\theta\right]}$$

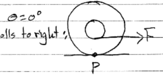
Eliminating f from these two boxed relations in (e), we find

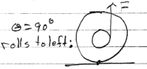
$$\boxed{\min \mu = \left(\frac{F}{1+b}\right)\frac{\left(\frac{r}{R} + b\cos\theta\right)}{(mg - F\sin\theta)}}$$

f) From the sign of a_x in solution (e), yoyo rolls to right for $\theta = 0°$ and indeed for any $\cos\theta < r/R$.

Common Mistake: "F torques counterclockwise so the yoyo rolls counterclockwise".
Rebutal: Although F does torque counterclockwise about C, f torques clockwise. Although $F > f$, careful analysis shows that $fR > Fr$ so the net torque about C is clockwise.
[Also, $F > f$ means the center of mass moves to right.]

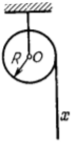
Alternative proof: Consider torques about point of contact P where only F contributes.
F provides clockwise torque about P (roll to right) for $\theta = 0°$ or any $\theta < \cos^{-1}(r/R)$.
[If $\theta > \cos^{-1}(r/R)$, torque about P is counterclockwise and yoyo rolls to left].



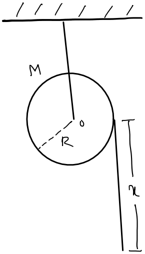


QUESTION 9 :

A uniform cylinder of radius R and mass M can rotate freely about a stationary horizontal axis O . A thin cord of length l and mass m is wound on the cylinder in a single layer. Find the angular acceleration of the cylinder as a function of the length x of the hanging part of the cord. The wound part of the cord is supposed to have its centre of gravity on the cylinder axis.



SOLUTION :



[This is a slightly advanced problem.]

$\tau = \frac{dL}{dt}$ by definition.

Let us look at the torque in our system. Torque about O is produced by the gravitational force acting on the part of the cord that is hanging down.

Mass of cord of length $x = \frac{mx}{l}$

$\therefore F_g = \frac{mx}{l} g$

and $\tau = \frac{mgx}{l} R$.

$L = (L_{cylinder} + L_{wrapped\ cord} + L_{hanging\ cord})$

$$= \frac{MR^2}{2}\omega + \frac{m(l-x)R^2\omega}{l} + \frac{mx}{l}vR$$

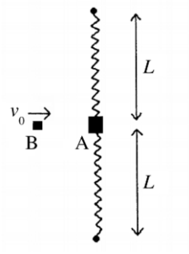
$$= \frac{MR^2}{2}\omega + \frac{m(l-x)R^2\omega}{l} + \frac{mxR^2\omega}{l}$$

$$= \frac{MR^2}{2}\omega + mR^2\omega$$

$$\frac{dL}{dt} = \left(\frac{MR^2}{2} + mR^2\right)\alpha = \tau$$

$$\Rightarrow \alpha = \frac{mgxR}{l\left(\frac{MR^2}{2} + mR^2\right)} = \frac{2mgx}{RL(M+2m)} \; //$$

QUESTION 10:



Block A has mass m and is attached to two identical springs each with spring constant k and each stretched to length L from equilibrium length 0. This apparatus sits on the surface of a frictionless tabletop, and the ends of the springs are attached to the tabletop. Block B has mass $m/2$ and slides with speed v_0 towards block A (which is at rest prior to the collision) as indicated in the diagram. The blocks undergo an inelastic collision, but they do not stick together. Immediately after the collision, block B is at rest. You may neglect the sizes of the blocks.

(a) (2 pt) What is the velocity v_f of block A immediately after the collision?

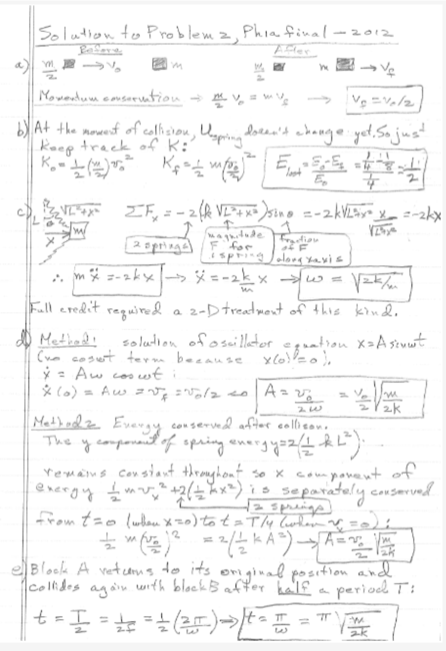
(b) (2 pt) What fraction of the initial mechanical energy is lost in the collision? (Be sure to express your answer as the *fraction* lost rather than the absolute amount lost.)

(c) (2 pts) After the initial collision, block A undergoes simple harmonic motion. What is the angular frequency ω of this motion?

(d) (2 pts) What is the amplitude of the simple harmonic motion after the initial collision?

(e) (2 pts) How long after the initial collision will the two blocks collide again?

SOLUTION:



Solution to Problem 2, Ph1a final — 2012

Before After

a) $\frac{m}{2} \rightarrow v_0$ m $\frac{m}{2}$ $m \rightarrow v_f$

Momentum conservation $\rightarrow \frac{m}{2} v_0 = m v_f \longrightarrow \boxed{v_f = v_0/2}$

b) At the moment of collision, U_{spring} doesn't change yet. So just keep track of K:

$K_0 = \frac{1}{2}\left(\frac{m}{2}\right)v_0^2$ $K_f = \frac{1}{2} m \left(\frac{v_0}{2}\right)^2$ $\boxed{E_{lost} = \frac{E_0 - E_f}{E_0} = \frac{\frac{1}{4} - \frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}}$

c) L, θ, $\sqrt{L^2+x^2}$, m, x

$\sum F_x = -2\left(k\sqrt{L^2+x^2}\right)\sin\theta = -2k\sqrt{L^2+x^2}\,\frac{x}{\sqrt{L^2+x^2}} = -2kx$

2 springs magnitude F for 1 spring fraction of F along x axis

$\therefore \boxed{m\ddot{x} = -2kx} \longrightarrow \ddot{x} = -2\frac{k}{m}x \longrightarrow \boxed{\omega = \sqrt{2k/m}}$

Full credit required a 2-D treatment of this kind.

d) Method 1: solution of oscillator equation $x = A\sin\omega t$ (no $\cos\omega t$ term because $x(0) = 0$).

$\dot{x} = A\omega\cos\omega t$;

$\dot{x}(0) = A\omega = v_f = v_0/2$ so $\boxed{A = \frac{v_0}{2\omega} = \frac{v_0}{2}\sqrt{\frac{m}{2k}}}$

Method 2 Energy conserved after collison.

The y component of spring energy $= 2\left(\frac{1}{2}kL^2\right)$;

remains constant throughout so x component of energy $\frac{1}{2}mv_x^2 + 2\left(\frac{1}{2}kx^2\right)$ is separately conserved

2 springs

from $t = 0$ (when $x = 0$) to $t = T/4$ (when $v_x = 0$):

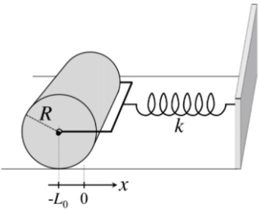
$\frac{1}{2}m\left(\frac{v_0}{2}\right)^2 = 2\left(\frac{1}{2}kA^2\right) \longrightarrow \boxed{A = \frac{v_0}{2}\sqrt{\frac{m}{2k}}}$

e) Block A returns to its original position and collides again with block B after half a period T:

$t = \frac{T}{2} = \frac{1}{2f} = \frac{1}{2}\left(\frac{2\pi}{\omega}\right) \Rightarrow \boxed{t = \frac{\pi}{\omega} = \pi\sqrt{\frac{m}{2k}}}$

QUESTION 11 :

A solid cylinder of mass M, radius R, and uniform density ρ is attached to a horizontal massless spring with spring constant k. The cylinder can roll without slipping along a horizontal surface as shown above. The system is released from rest at a position in which the spring is stretched a distance L_0 relative to its equilibrium length. *(Note: not all the quantities listed in this paragraph are relevant to the questions below.)*



(a) (2 pt) Using Newton's second law, write an equation for the horizontal acceleration of the center of mass of the cylinder. You can label the horizontal force due to friction as F_f.

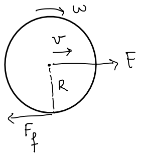
(b) (2 pt) Write an equation for the angular acceleration of the cylinder about its center of mass, again using F_f if it is required.

(c) (4 pt) Using your equations in (a) and (b), find the acceleration of the cylinder's center of mass in terms of the displacement x and quantities given in the opening paragraph. Remember that the cylinder rolls without slipping.

(d) (2 pt) Explain why the answer in (c) implies that the center of mass undergoes simple harmonic motion. What is the period of this oscillatory motion.

SOLUTION 11 :

a)



Using Newton's second law,

$$F - F_f = Ma.$$

$$\Rightarrow \quad a = \frac{F - F_f}{M}$$

where $F = -kx$.

b) $\tau = F_f R = I\alpha$

$$\therefore \alpha = \frac{F_f R}{I} = \frac{2F_f}{MR}$$

Since $I = \frac{1}{2}MR^2$ for a cylinder.

c) Since the cylinder is rolling without slipping,

$$v = \omega R \Rightarrow \quad a = \alpha R$$

$$\therefore \alpha = \frac{a}{R} = \frac{2F_f}{MR}$$

or $F_f = \dfrac{Ma}{2}$

Substituting this in the equation in part (a)

gives $a = \dfrac{F}{M} - \dfrac{a}{2}$

or $\dfrac{3a}{2} = \dfrac{-kx}{M} \quad \Rightarrow \quad a = -\dfrac{2kx}{3M}.$

d) $a = \dfrac{2kx}{3M} \Rightarrow \dfrac{d^2x}{dt^2} = \dfrac{-2kx}{3M}$. This is the equation for a simple harmonic oscillator with $\omega = \sqrt{\dfrac{2k}{3M}}$.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3M}{2k}} = \pi\sqrt{\frac{6M}{k}} \,//$$

QUESTION 12 :

A satellite is in a circular earth orbit of altitude 400 km. Determine the new perigee and apogee altitudes if the satellite on-board engine

(a) increases the speed of the satellite in the flight direction by 240 m/s;

(b) gives the satellite a radial (outward) component of velocity of 240 m/s.

SOLUTION :

For a circular orbit,

$$\frac{mv^2}{R_0} = \frac{GMm}{R_0^2}$$ (Force due to gravity is the centripetal force)

$$\therefore v = \sqrt{\frac{GM}{R_0}}$$

a) If velocity increases in the direction of flight, the new velocity is $\left(240 \text{ m/s} + \sqrt{\frac{GM}{R_0}}\right)$.

This point is either the perigee or apogee of the new orbit since the is no radial velocity.

We have $E = \frac{1}{2}mv^2 - \frac{GMm}{R_0} = -\frac{GMm}{2a}$

$$\Rightarrow \quad a = -\frac{GM/2}{\frac{1}{2}v^2 - \frac{GM}{R_0}} = \frac{GM/2}{\frac{GM}{R_0} - \frac{1}{2}\left(\sqrt{\frac{GM}{R_0}} + 240\right)^2} \approx 7240 \text{ km}$$

Now, $r_{min} = a(1-\varepsilon)$, $r_{max} = a(1+\varepsilon)$.

Radius of earth ~ 6380 km.

$\therefore R_0 \approx 6780$ km.

If we take this to be the apogee, $\varepsilon < 0$.

$\therefore r_{min} = 6780$ km.

$$\Rightarrow \quad \varepsilon = 1 - \frac{6780}{7240}$$

$$r_{max} = 7240\left(2 - \frac{6780}{7240}\right) \approx 7700 \text{ km}.$$

$\therefore$ Perigee is at an altitude of 400 km and apogee is at an altitude of 1320 km.

(Alternatively, the problem can be solved using energy and angular momentum conservation at the perigee and apogee.)

b) If velocity increases radially, this point is not the perigee or apogee.

$$E = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_0^2 - \frac{GMm}{R}$$

$$= \frac{-m}{2}(7655)^2$$

$$L = mv_0R$$

$$= m\ 7660 \times 6781 \times 10^3 .$$

$$d = \frac{L^2}{GMm^2} = \frac{V_0^2R^2}{GM} = R = 6781 \text{ km.}$$

and $$\varepsilon = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$$

$$= \left(1 + \frac{2ER}{GMm}\right)^{1/2}$$

$$= 0.036 .$$

$$\therefore r_{min} = \frac{d}{1+\varepsilon} \quad , \quad r_{max} = \frac{d}{1-\varepsilon} .$$

$$= 6545 \text{ km} \qquad = 7034 \text{ km.}$$

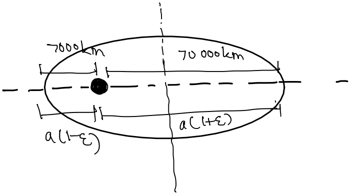
∴ perigee altitude = 164 km,

apogee altitude = 653 km.

QUESTION 13 :

An unmanned satellite orbits the earth with a perigee radius of 7000 km and an apogee radius of 70 000 km. Calculate

(a) the eccentricity of the orbit;
(b) the semimajor axis of the orbit (km);
(c) the period of the orbit (hours);



(a and b) $a(1-\varepsilon) = 7000$

$a(1+\varepsilon) = 70000$

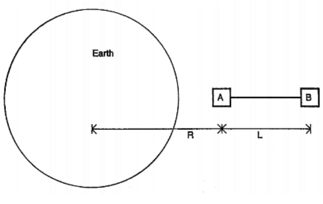
$\Rightarrow \quad \frac{1+\varepsilon}{1-\varepsilon} = 10$

$\Rightarrow \quad \varepsilon = 3/11$

and $a = 38500$ km.

c) $T = \sqrt{\frac{4\pi^2}{GM}}\, a^{3/2} = 20.89$ hours

QUESTION 14 :



Two satellites, A and B, orbit the Earth as shown. Satellite A is at distance R from the center of the Earth. Satellite B is attached to satellite A by a tether of length L. The Earth has mass M. The two satellites are identical to each other and each has mass m. The satellites are given the same angular velocity ω so that they remain in circular orbits around the Earth.

You may assume that $m \ll M$ – that is, you may make the approximation that the satellites orbit around the center of the Earth. However you may not neglect the mutual gravitational attraction of the two satellites.

(a) (2 pts) Draw the free-body force diagram for each satellite.

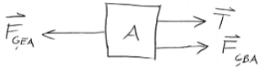
(b) (3 pts) Find an expression for ω in terms of M, m, R, L, and fundamental constants.

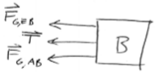
(c) (3 pts) What is the tension T in the tether?

(d) (2 pts) Is there a minimum value of L for which the tether remains in tension? Explain. (You do not need to solve for this minimum value.)

SOLUTION :

(a) (2 pts)





(b) (3 pts) We can approximately ignore the effects of the Earth orbiting around the Sun. Also, I will work in an inertial reference frame which is not rotating with the satellites. Using the FBDs from above, I can write down the net force equation in the radial direction for each satellite.

$$\Sigma F_{A,r} = ma_{A,r}$$

$$\frac{-GMm}{R^2} + T + \frac{Gm^2}{L^2} = -m\omega^2 R.$$

$$\Sigma F_{B,r} = ma_{B,r}$$

$$\frac{-GMm}{(R+L)^2} - T - \frac{Gm^2}{L^2} = -m\omega^2(R+L).$$

Note that the satellites are moving in circular orbits. This is why the accelerations of the satellites in the radial direction are given by $\omega^2 r$, where r is the distance of the given satellite from the center of the earth. Adding these equations together gives

$$-GMm\left[\frac{1}{R^2} + \frac{1}{(R+L)^2}\right] = -m\omega^2(2R+L).$$

Solving for ω yields

$$\omega = \sqrt{\frac{GM}{2R+L}\left[\frac{1}{R^2} + \frac{1}{(R+L)^2}\right]}.$$

(c) (3 pts) Solving for T in the net force equation for satellite A and substituting in ω gives us

$$T = -m\omega^2 R + \frac{GMm}{R^2} - \frac{Gm^2}{L^2} = Gm\left\{\frac{M}{R^2} - \frac{m}{L^2} - \frac{M}{2+L/R}\left[\frac{1}{R^2} + \frac{1}{(R+L)^2}\right]\right\}.$$

(d) (2 pts) Yes. As the satellites approach each other (i.e. as L decreases), the gravitational attraction between them will grow, and eventually T in the above expression will go to zero. At this point, the gravitational attraction between the two satellites and from the Earth on each of the satellites provides the correct centripetal force, and the rope is limp. If we tried to move the satellites even close together, we would need to place a solid bar between them to keep them from falling into each other.